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Simple-ribbon fusions and primeness of links [1]

by

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Abstract

In [KST16], we introduced a special kind of fusion, (elementary) simple-ribbon fusion, for knots and links, and in [KST18], we studied the primeness of knots obtained by an elementary simple-ribbon fusion. In this paper, we study the case for links.

Keywords; knots, links, primeness

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1. Introduction

Knots and links are assumed to be ordered and oriented, and they are considered up to ambient isotopy in an oriented 3-sphere S^3. Throughout this paper links are assumed to have at least 2 components, and thus a knot is not a link. In [KST16], we introduced special types of fusions, so called simple-ribbon fusions. Here we only define an elementary simple-ribbon fusion. Refer [KST16] for a general simple-ribbon fusion, which can be realized by elementary simple-ribbon fusions.

A (m-)*ribbon fusion* on a link ℓ is an m-*fusion* ([AK96, Definition 13.1.1]) on the split union of ℓ and an m-component trivial link $\mathcal{O}$ such that each component of $\mathcal{O}$ is attached to a component of ℓ by a single band. Note that any knot obtained from the trivial knot by a finite sequence of ribbon fusions is a *ribbon knot* ([AK96, Definition 13.1.9]), and that any ribbon knot can be obtained from the trivial knot by ribbon fusions.

Let ℓ be a link and $\mathcal{O} = O_1 \cup \cdots \cup O_m$ the m-component trivial link which is split from ℓ. Let $\mathcal{D} = D_1 \cup \cdots \cup D_m$ be a disjoint union of non-singular disks with $\partial D_i = O_i$ and $D_i \cap \ell = \emptyset$ $(i = 1, \cdots, m)$, and let $\mathcal{B} = B_1 \cup \cdots \cup B_m$ be a disjoint union of disks for an m-fusion, called *bands*, on the split union of ℓ and $\mathcal{O}$ satisfying the following:

(i) $B_i \cap \ell = \partial B_i \cap \ell = \{$ a single arc $\}$;

(ii) $B_i \cap \mathcal{O} = \partial B_i \cap O_i = \{$ a single arc $\}$; and

(iii) $B_i \cap \operatorname{int} \mathcal{D} = B_i \cap \operatorname{int} D_{i+1} = \{$ a single arc of ribbon type $\}$, where the indices are considered modulo m.

Let L be a link obtained from the split union of ℓ and $\mathcal{O}$ by the m-fusion along $\mathcal{B}$, i.e., $L = (\ell \cup \mathcal{O} \cup \partial\mathcal{B}) - \operatorname{int}(\mathcal{B} \cap \ell) - \operatorname{int}(\mathcal{B} \cap \mathcal{O})$. Then we say that L is obtained from ℓ by an *elementary simple-ribbon fusion*, or *SR-fusion* for short, *of type* m (*with respect to* $\mathcal{D} \cup \mathcal{B}$).

An elementary SR-fusion is *trivial* if $\mathcal{O}$ bounds mutually disjoint non-singular disks $\cup\Delta_i$ such that $\partial\Delta_i = O_i$ and that $\operatorname{int}\Delta_i$ does not intersect with $L \cup \mathcal{B}$ for each i $(1 \leq i \leq m)$. Here note that $\cup\Delta_i$ may intersect with $\operatorname{int}\mathcal{D}$. Since L is ambient isotopic to ℓ through $(\cup\Delta_i) \cup \mathcal{B}$, we know that any trivial SR-fusion does not change the link type. It is easy to see that an elementary SR-fusion is trivial if and only if there is an j $(1 \leq j \leq m)$ such that O_j bounds a non-singular disk whose interior does not intersect with $L \cup \mathcal{B}$.

A non-singular 2-sphere Σ is called *a decomposing sphere of a link* L if Σ intersects with L transversally in two points. A decomposing sphere of L is called *trivial* if Σ bounds a 3-ball intersecting with L in a trivial arc. A link L is said to be *split* if there is a non-singular 2-sphere Ω in $S^3 - L$ such that $E_1 \cup E_2 = S^3$, $E_1 \cap E_2 = \Omega$, and L_i $(= L \cap E_i^3) \neq \emptyset$ $(i = 1, 2)$. A non-split link L is *prime* if any decomposing sphere for L is trivial. We remark here that the 2-component trivial link is the only split link which admits a non-trivial decomposing sphere, and also the only trivial link which admits a non-trivial decomposing sphere.

A non-trivial SR-fusion on a link ℓ with respect to $\mathcal{D} \cup \mathcal{B}$ is *prime* if for any 2-sphere Σ which intersects with $\ell - \mathcal{B}$ transversally in two points and satisfies that $\Sigma \cap (\mathcal{D} \cup \mathcal{B}) = \emptyset$, Σ bounds a 3-ball H such that $H \cap (\mathcal{D} \cup \mathcal{B}) = \emptyset$ and that $H \cap \ell$ is a trivial arc. Then we showed the following in [KST18].

Theorem 1.1. ([KST18, Theorem 1.1]) *Let K be a knot obtained from a knot k by a prime elementary SR-fusion. If the type of the elementary SR-fusion is no less than* 3*, k is non-trivial, or K is neither $3_1 \sharp \overline{3_1}$ nor $4_1 \sharp 4_1$, then K is prime.*

The following is our main theorem.

Theorem 1.2. *Let L be a link obtained from a link ℓ by an elementary SR-fusion. If the SR-fusion is non-trivial and prime, then L is prime.*

Corollary 1.3. *Let L be a link obtained from a link ℓ by an elementary SR-fusion. If ℓ is a trivial link $\mathcal{O}$ and L is a non-split link, then L is prime.*

Proof. Since ℓ is a trivial link and L is a non-split link, L is not ambient isotopic to ℓ. Hence the elementary SR-fusion is not trivial by Theorem 1.1 of [KST16]. Next let Σ be a 2-sphere which intersects with $\ell - \mathcal{B}$ transversally in two points and satisfies that $\Sigma \cap (\mathcal{D} \cup \mathcal{B}) = \emptyset$. We may assume that Σ intersects with O^1 of $\ell = \mathcal{O}$. Let H be a 3-ball bounded by Σ such that $H \cap (\mathcal{D} \cup \mathcal{B}) = \emptyset$. Since genus of knot is additive under connected sum, $O^1 \cap H$ is a trivial arc. In fact, since $\mathcal{O}$ is a trivial link and L is non-split, $\ell \cap H = O^1 \cap H$. Hence the elementary SR-fusion is also prime, and thus L is prime by Theorem 1.2. $\square$

Corollary 1.4. *Let L be a link obtained from a link ℓ by an elementary SR-fusion. If ℓ is a non-split link and the SR-fusion is prime, then L is prime.*

Proof. It is sufficient to show that the SR-fusion is non-trivial. Assume otherwise. Since ℓ is a non-split link, the SR-fusion is of type 1 with respect to $D_1 \cup B_1$, and $O_1 = \partial D_1$ bounds a non-singular disk Δ such that $\mathrm{int}\Delta \cap (L \cup D_1 \cup B_1) = \emptyset$ by Theorem 1.2 of [KST17]. Let Σ be the 2-sphere $\Delta \cup D_1$. Push D_1 of Σ to the direction of $B_1 \cap \ell$ so to separate D_1 and ℓ by Σ. Then slide $\Sigma \cap B_1$ along B_1 to $B_1 \cap \ell$ and push $\Sigma \cap B_1$ so that Σ intersects with $L \cup D_1 \cup B_1$ in two points of $\ell - B_1$. Since ℓ is non-split, and thus ℓ is non-trivial, we can see that our elementary SR-fusion is not prime, which is a contradiction. $\square$

2. Proof of Theorem 1.2

Let L be a link obtained from a link ℓ by an elementary SR-fusion of type m with respect to $\mathcal{D} \cup \mathcal{B} = (D_1 \cup \cdots \cup D_m) \cup (B_1 \cup \cdots \cup B_m)$ and Σ a decomposing sphere for L. We may assume that each D_i is a plane disk ($1 \leq i \leq m$), and that Σ and $\mathcal{D} \cup \mathcal{B}$ intersects transversally.

Let $\dot{D}_i$ and $\dot{B}_i$ be disks and $f : \cup_i \left(\dot{D}_i \cup \dot{B}_i\right) \to S^3$ an immersion such that $f(\dot{D}_i) = D_i$ and $f(\dot{B}_i) = B_i$. We denote the arc of int $D_i \cap B_{i-1}$ by α_i and let $B_{i,1}$ and $B_{i,2}$ be the subdisks of B_i such that $B_{i,1} \cup B_{i,2} = B_i$, $B_{i,1} \cap B_{i,2} = \alpha_{i+1}$, and $B_{i,1} \cap \partial D_i \neq \emptyset$. Take a point b_i on int α_i, an arc β_i on $D_i \cup B_{i,1}$ so that $b_i \cap (\alpha_i \cup \alpha_{i+1}) = \partial\beta_i = b_i \cup b_{i+1}$, and orient the arc β_i from b_{i+1} to b_i ($i = 1, \ldots, m$) (see Figure 1). Then $\beta = \cup_i \beta_i$ is an oriented simple loop and we call β an *attendant knot* of $\mathcal{D} \cup \mathcal{B}$. Moreover, we denote the pre-images of α_i (resp. b_i) on $\dot{D}_i$ and $\dot{B}_{i-1}$ by $\dot{\alpha}_i$ and $\ddot{\alpha}_i$ (resp. $\dot{b}_i$ and $\ddot{b}_i$), respectively.

The set $\mathcal{S}_i$ of the pre-images on $\dot{D}_i \cup \dot{B}_i$ of the intersections of Σ and $D_i \cup B_i$ consists of arcs and loops which are mutually disjoint and simple. Let $\mathcal{S} = \cup_i \mathcal{S}_i$. Define the *complexity* of Σ as the lexicographically ordered set (s_1, s_2, s_3), where s_1 (resp. s_2) is the number of arcs (resp. loops) of $\mathcal{S}$ and s_3 is the number of triple points of $(\mathcal{D} \cup \mathcal{B}) \cup \Sigma$. An arc of $\mathcal{S}_i$ is *standard* if the

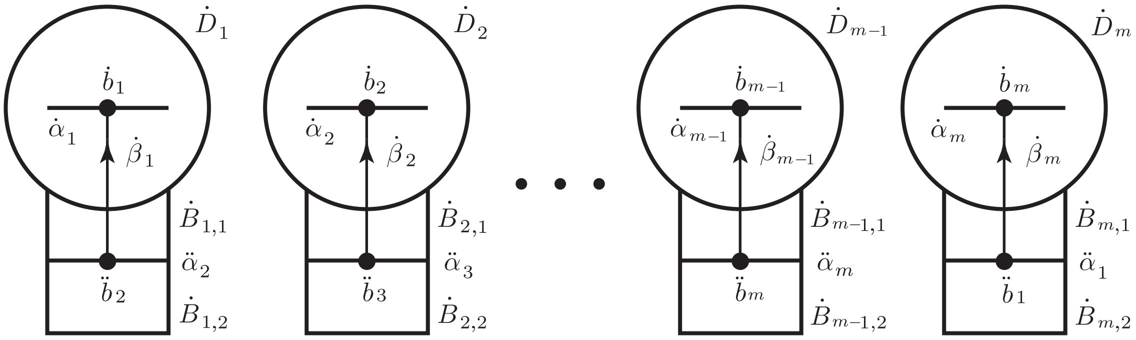

FIGURE 1

arc has one end on $\partial\dot{D}_i - \partial\dot{B}_i$ and the other end on the pre-image of $\partial B_i \cap \ell$, and intersects with each of $\dot{\alpha}_i$ and $\ddot{\alpha}_{i+1}$ exactly once (see type 3b of Figure 5). We say that Σ is in a *standard position* if $\mathcal{S}$ consists of only standard arcs.

Lemma 2.1. *Let L be a link obtained from a link ℓ by a non-trivial elementary SR-fusion with respect to $\mathcal{D} \cup \mathcal{B}$. If Σ has the minimal complexity among all the non-trivial decomposing sphere for L and satisfies that $\Sigma \cap (\mathcal{D} \cup \mathcal{B}) \neq \emptyset$, then Σ is in a standard position.*

Proof. Since $\Sigma \cap (\mathcal{D} \cup \mathcal{B}) \neq \emptyset$, we have that $\mathcal{S} \neq \emptyset$.

Claim 2.2. *$\mathcal{S}_i$ does not have a loop which bounds a disk on $\dot{D}_i \cup \dot{B}_i$ intersecting with neither $\dot{\alpha}_i$ nor $\ddot{\alpha}_{i+1}$ for each i.*

Proof. Assume otherwise. Take an innermost one $\dot{\rho}$ from such loops on $\dot{D}_i \cup \dot{B}_i$ and let δ be the disk bounded by $\rho = f(\dot{\rho})$ on $D_i \cup B_i$. Replace a neighborhood of ρ in Σ with two parallel copies of δ (see Figure 2). We obtain two spheres Σ_1 and Σ_2, where we may assume that $\Sigma_1 \cap L$ consists of two points and $\Sigma_2 \cap L = \emptyset$. Then Σ_1 is another non-trivial decomposing sphere for L with less complexity than that of Σ, which contradicts that Σ has the minimal complexity. □

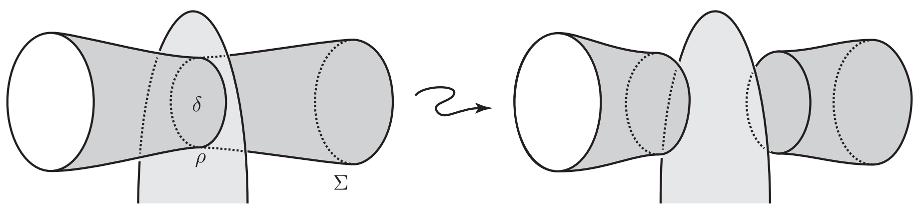

FIGURE 2. surgery on Σ with respect to δ

Claim 2.3. *None of the elements of $\mathcal{S}_i$ has a subarc which bounds a disk on $\dot{D}_i \cup \dot{B}_i$ with a subarc of* int $\dot{\alpha}_i$ *or* int $\ddot{\alpha}_{i+1}$ *whose interior is disjoint from both of $\dot{\alpha}_i$ and $\ddot{\alpha}_{i+1}$.*

Proof. Assume otherwise and take an innermost one from such subarcs, i.e., it bounds a disk $\dot{\delta}$ on $\dot{D}_i \cup \dot{B}_i$ with a subarc of int $\dot{\alpha}_i$ (resp. int $\ddot{\alpha}_{i+1}$) whose interior does not contain any other such subarcs. Since $\dot{\delta}$ does not contain any loops from Claim 2.2, we can deform $\partial(\delta \times I) \cap \Sigma$ of Σ to the closure δ' of $\partial(\delta \times I) - \Sigma$ along $\delta \times I$ as illustrated in Figure 3 and push δ' of Σ out of B_{i-1} (resp. D_{i+1}) to eliminate the two triple points, which contradicts that Σ has the minimal complexity. □

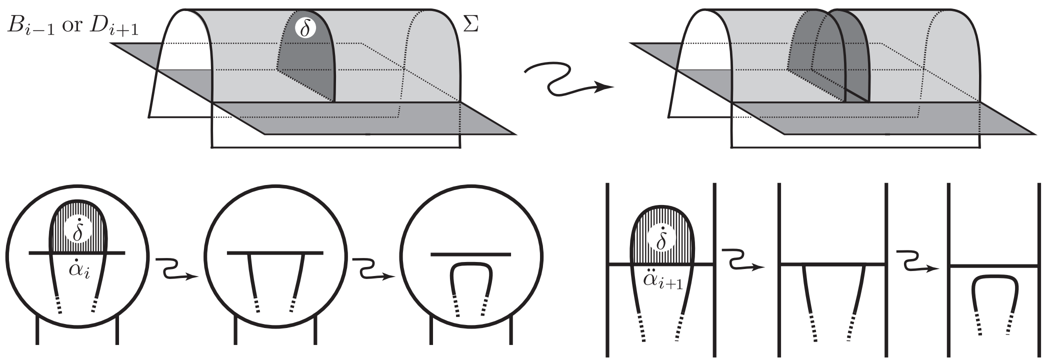

FIGURE 3. eliminating triple points

Claim 2.4. *$\mathcal{S}$ has no loops.*

Proof. By the above two claims, we may assume that each loop of $\mathcal{S}_i$ is on $\dot{D}_i$, and bounds a disk on $\dot{D}_i$ containing $\dot{\alpha}_i$ or intersects with $\dot{\alpha}_i$ in one point. Let $\dot{\rho}$ be a loop of $\mathcal{S}_i$.

Assume that $\dot{\rho}$ bounds a disk $\dot{\delta}$ on $\dot{D}_i$ containing $\dot{\alpha}_i$. Since $\delta = f(\dot{\delta})$ intersects with L in two points of $\partial\alpha_i$, one component of $\Sigma - \rho$ intersects with L in two points and the other component δ' does not intersect with L. Thus we can slide $L \cap \partial(D_i \cup B_i)$ onto $\ell \cap B_i$ along $((D_i - \delta) \cup \delta') \cup B_i$, which induces that the SR-fusion is trivial by Theorem 1.1 of [KST16], which contradicts the assumption.

If $\mathcal{S}_i$ has a loop on $\dot{D}_i$ which intersects with $\dot{\alpha}_i$ in one point, then take an innermost one $\dot{\rho}$ on $\dot{D}_i$ and let δ be the disk bounded by $\rho = f(\dot{\rho})$ on D_i. Replace a neighborhood of ρ in Σ with two parallel copies of δ as illustrated in Figure 4. Then we have two spheres Σ_1 and Σ_2 and at least one sphere, say Σ_1 is a non-trivial decomposing sphere for L, whose complexity is less than that of Σ. This contradicts that Σ has the minimal complexity. Thus we complete the proof. □

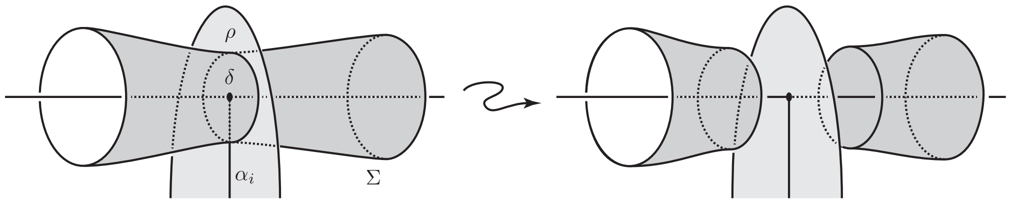

FIGURE 4. surgery on Σ along δ

Therefore each $\mathcal{S}_i$ has only arcs. We may assume that the end points of the image of each arc by f are on $(\partial D_i - \partial B_i) \cup (\partial B_i \cap \ell)$ by isotoping Σ so that the end point on $\partial B_{i,1}$ (resp. $\partial B_{i,2}$) moves onto $\partial D_i - \partial B_i$ (resp. $\partial B_i \cap \ell$) if necessary. Then each arc $\dot{\gamma}$ is one of the following 8 types.

Type 1: the both two end points are on $\partial\dot{D}_i - \partial\dot{B}_i$. Let $\dot{\delta}$ be the subdisk of $\dot{D}_i$ bounded by $\dot{\gamma}$ with a subarc $\dot{\zeta}$ of $\partial\dot{D}_i - \partial\dot{B}_i$. We have three cases that $\dot{\delta} \cap \dot{\alpha}_i = \emptyset$ (Type 1a), $\dot{\gamma}$ intersects with int $\dot{\alpha}_i$ in one point (Type 1b), or $\dot{\delta}$ contains $\dot{\alpha}_i$ (Type 1c).

Type 2: the both two end points are on the pre-image of $\partial B_i \cap \ell$. Let $\dot{\delta}$ be the subdisk of $\dot{D}_i \cup \dot{B}_i$ bounded by $\dot{\gamma}$ with a subarc $\dot{\zeta}$ of the pre-image of $\partial B_i - \ell$. We have three cases that $\dot{\delta}$ is in $\dot{B}_{i,2}$ (Type 2a), $\dot{\gamma}$ intersects with int $\dot{\alpha}_i$ in one point (Type 2b), or $\dot{\delta}$ contains $\dot{\alpha}_i$ (Type 2c).

Type 3: one end point is on $\partial\dot{D}_i - \partial\dot{B}_i$ and the other end point is on the pre-image of $\partial B_i \cap \ell$. $\dot{\gamma}$ does not intersect with $\dot{\alpha}_i$ (Type 3a) or $\dot{\gamma}$ intersects with $\dot{\alpha}_i$ in one point (Type 3b).

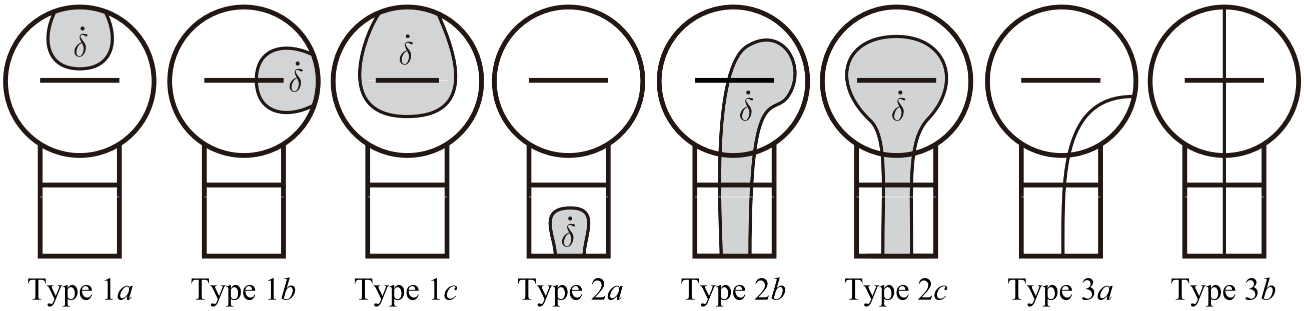

Figure 5

Let H be the 3-ball bounded by Σ which contains δ in the first 6 cases. Note that there does not exist an arc of type 1a, since otherwise $L \cap H = \zeta$ is a trivial arc, which contradicts that Σ is a non-trivial decomposing sphere. In addition there does not exist an arc of type 2a, since otherwise we can eliminate it by pushing Σ out of B_i.

Assume that $\mathcal{S}$ contains an arc of type 1b and that $\dot{D}_h \cup \dot{B}_h$ contains such an arc $\dot{\gamma}$. Since Σ intersects with L in two points, any arc of $\mathcal{S}$ other than $\dot{\gamma}$ has type 2b or 2c. Since $\alpha_h \cap \Sigma \neq \emptyset$, $\dot{D}_{h-1} \cup \dot{B}_{h-1}$ contains an arc of type 2b or 2c. Thus $\dot{D}_h \cup \dot{B}_h$ contains an arc of type 2b. Then inductively from $\dot{D}_{h+1} \cup \dot{B}_{h+1}$ we can see that $\dot{D}_i \cup \dot{B}_i$ contains an arc of type 2b for any i ($1 \leq i \leq m$). Hence we know that $\dot{D}_h \cup \dot{B}_h$ contains one arc of type 1b and arcs of type 2b, and $\dot{D}_i \cup \dot{B}_i$ ($i \neq h$) contains at least one arc of type 2b and possibly arcs of type 2c. Now consider the number $\sharp\,(\mathcal{S} \cap \dot{\alpha}_i)$ of intersections of $\mathcal{S}$ and $\dot{\alpha}_i$ ($1 \leq i \leq m$). Since $f(\dot{\alpha}_i) = f(\ddot{\alpha}_i)$, we have that $\sharp\,(\mathcal{S} \cap \dot{\alpha}_i) = \sharp\,(\mathcal{S} \cap \ddot{\alpha}_i)$. Thus we have the following for h and i ($1 \leq i \leq m, i \neq h$).

$$\sharp\,(\mathcal{S} \cap \dot{\alpha}_{h+1}) = \sharp\,(\mathcal{S} \cap \ddot{\alpha}_{h+1}) \geq \sharp\,(\mathcal{S} \cap \dot{\alpha}_h),$$
$$\sharp\,(\mathcal{S} \cap \dot{\alpha}_{i+1}) = \sharp\,(\mathcal{S} \cap \ddot{\alpha}_{i+1}) > \sharp\,(\mathcal{S} \cap \dot{\alpha}_i).$$

Here note that $\sharp\,(\mathcal{S} \cap \dot{\alpha}_{m+1}) = \sharp\,(\mathcal{S} \cap \dot{\alpha}_1)$, since we consider the lower index modulo m. Hence we have that $m = h = 1$, since otherwise we have that $\sharp\,(\mathcal{S} \cap \dot{\alpha}_{m+1}) > \sharp\,(\mathcal{S} \cap \dot{\alpha}_1)$. Thus we have two cases for $\dot{D}_1 - \partial\dot{B}_1$ as illustrated in Figure 6 depending how $f(\dot{D}_1)$ and $f(\dot{B}_1)$ intersect. Let $\dot{p}$ be the boundary point of $\dot{\alpha}_1$ in $\dot{\delta}$ containing the arc of type 1b and take an arc $\dot{\eta}$ connecting $\dot{p}$ and $\ddot{p}$ which is the boundary point of $\ddot{a}_1$ and a pre-image of $f(\dot{p})$ as illustrated in Figure 6. However then, the loop $f(\dot{\eta})$ intersects Σ only once, which is impossible. Hence there does not exist an arc of type 1b.

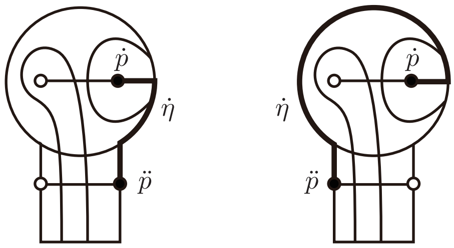

Figure 6

Now assume that $\mathcal{S}$ contains an arc of type 1c and that $\dot{D}_h \cup \dot{B}_h$ contains such an arc $\dot{\gamma}$. Since Σ intersects with L in two points, any arc of $\mathcal{S}$ other than $\dot{\gamma}$ has type 2b or 2c. However since $\sharp\,(\mathcal{S} \cap \dot{\alpha}_h) = 0$, $\dot{D}_{h-1} \cup \dot{B}_{h-1}$ contains neither an arc of type 2b nor an arc of type 2c. Hence $\dot{D}_{h-1} \cup \dot{B}_{h-1}$ contains no arcs of $\mathcal{S}$ and inductively we can see that $\dot{D}_i \cup \dot{B}_i$ contains no arcs of $\mathcal{S}$ for any i $(1 \leq i \leq m, i \neq h)$. Then an attendant knot of $\mathcal{D} \cup \mathcal{B}$ intersects with Σ only once, which is impossible. Hence there does not exist an arc of type 1c.

Hence we know that any arc has type 3b by considering the number $\sharp\,(\mathcal{S} \cap \dot{\alpha}_i)$ of intersections of $\mathcal{S}$ and $\dot{\alpha}_i$ $(1 \leq i \leq m)$. Hence Σ is in a standard position. $\square$

Lemma 2.5. *A link L obtained from a link ℓ by a prime SR-fusion is non-split.*

Proof. Assume that L is split and let Σ be a splitting sphere for L. Take a component ℓ_1 of ℓ such that $\ell_1 \cap \mathcal{B} \neq \emptyset$ and a point p of $\ell_1 - \mathcal{B}$. Let H be a neighborhood of p such that $H \cap (\ell \cup \mathcal{D} \cup \mathcal{B})$ is a trivial arc. Then take an arc γ in $S^3 - (\ell \cup \mathcal{D} \cup \mathcal{B})$ connecting a point on ∂H and a point of Σ. Let V be a neighborhood of γ in the closure of a component of $S^3 - \partial H - \partial \Sigma$. Then $\Sigma' = \partial H \cup \Sigma \cup \partial V - \mathrm{int}(\partial H \cap \partial V) - \mathrm{int}(\Sigma \cap \partial V)$ is a sphere which bounds a 3-ball H' such that $H' \cap (\mathcal{D} \cup \mathcal{B}) = \emptyset$ and $H' \cap \ell$ is not a trivial arc, which contradicts that the SR-fusion is prime. $\square$

Proof of Theorem 1.2. Assume that L is not prime and let Σ be a non-trivial decomposing sphere for L which has the minimal complexity among all the non-trivial decomposing sphere for L. Note that $\Sigma \cap (\mathcal{D} \cup \mathcal{B}) \neq \emptyset$, since the SR-fusion is prime and Σ is a non-trivial decomposing sphere for L. Hence Σ is in a standard position by Lemma 2.1.

Therefore, each $\mathcal{S}_i$ consists of the same non-zero number of standard arcs $(1 \leq i \leq m)$. Since $\Sigma \cap K$ consists of just two points, we have the following three cases:

Case 1 : $m = 1$ and $\mathcal{S}_1$ consists of one standard arc.

Case $2a$: $m = 1$ and $\mathcal{S}_1$ consists of two standard arcs.

Case $2b$: $m = 2$ and $\mathcal{S}_i$ consists of one standard arc $(i = 1, 2)$.

Let E_1 and E_2 be 3-balls such that $E_1 \cup E_2 = S^3$, $E_1 \cap E_2 = \Sigma$. In Case 1 and $2b$, take a neighborhood F_i of $(D_1 \cup B_1)$ in E_i, and let Σ_i be $\Sigma \cup \partial F_i - \mathrm{int}(\Sigma \cap F_i)$. Then Σ_i is a 2-sphere which intersects with $\ell - \mathcal{B}$ transversally in two points and satisfies that $\Sigma_i \cap (\mathcal{D} \cup \mathcal{B}) = \emptyset$, and thus Σ_i bounds a 3-ball H_i such that $H_i \cap (\mathcal{D} \cup \mathcal{B}) = \emptyset$ and that $H_i \cap \ell$ is a trivial arc, since the SR-fusion is prime. Hence ℓ is a knot, which contradicts that ℓ is a link.

In Case $2a$, assume that E_1 contains $\partial \alpha_1$. Similarly to the above case, take a neighborhood F_1 of $(D_1 \cup B_1)$ in E_1, and let Σ_1 be $\Sigma \cup \partial F_1 - \mathrm{int}(\Sigma \cap F_1)$. Then Σ_1 is a 2-sphere which intersects with $\ell - \mathcal{B}$ transversally in two points and satisfies that $\Sigma_1 \cap (\mathcal{D} \cup \mathcal{B}) = \emptyset$, and thus Σ_1 bounds a 3-ball H_1 such that $H_1 \cap (\mathcal{D} \cup \mathcal{B}) = \emptyset$ and that $H_1 \cap \ell$ is a trivial arc, since the SR-fusion is prime.

Now take a look at E_2. Note that $E_2 - (\mathcal{D} \cup \mathcal{B})$ consists of the interior of a solid torus V and the interior of a 3-ball as illustrated in Figure 7. Since L is non-split by Lemma 2.5 and ℓ is a link, E_2 contains a component ℓ_1 of ℓ in V which is homotopic to a longitude g of ∂V. Then isotop ℓ_1 to the direction of g and push ℓ_1 out of E_2 into E_1, moreover into H_1. However then, Σ_1 bounds a 3-ball H_1 such that $H_1 \cap (\mathcal{D} \cup \mathcal{B}) = \emptyset$ and that $H_1 \cap \ell$ is not a trivial arc, which contradicts that the SR-fusion is prime.

□

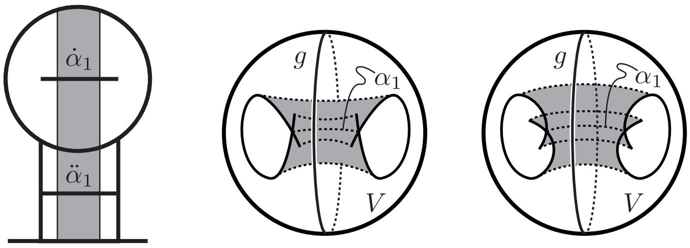

FIGURE 7

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