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On Local Moves for Links of Genus Zero, II

by

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Abstract

In the previous paper, [5], we studied that the links of genus 0 were self # -or self pass-equivalent to trivial links under some conditions.

In this paper, we study that two homotopic links of genus 0 are self # -or self pass-equivalent under what kinds of conditions.

1 Introduction.

In the previous paper, [5], we studied that the links of genus 0 were self $\sharp$- or self pass-equivalent to trivial links under some conditions. The definitions of self $\sharp$- and self pass-equivalence of links, see [3], [5], [6]

In this paper, we study that two homotopic links of genus 0 are self $\sharp$- or self pass-equivalent under what kind of conditions.

We deal the self $\sharp$- and self pass-equivalence for homotopic links as the boundaries of surfaces of genus 0 in section 2 and we do for links as the boundaries of mutually disjoint annuli and prove Theorems 2.3, 2.5 and 3.1 respectively.

As applications of these theorems, we attempt the classifications of 2-,3- and 4-component homotopic links of genus 0 and show Corollaries 2.6 and 3.4.

2 General case: Links as the boundaries of surfaces of genus 0.

An n-component link $\ell = k_1 \cup \ldots \cup k_n$ in R^3 is said to be *proper* if the linking number, denoted by $Link(k_i, \ell - k_i)(=\sum_{j \neq i} Link(k_i, k_j))$, is even for each $i(=1, \ldots, n)$. Especially ℓ is said to be *purely proper* if $Link(k_i, k_j)$ is even for $1 \leq i \neq j \leq n$. If ℓ is proper, the Arf invariant, [4], of ℓ, denoted by $\varphi(\ell)$, is well-defined and so the reduced Arf invariant, denoted by $\bar{\varphi}(\ell)(\equiv \varphi(\ell) - \Phi(\ell) \bmod 2)$ of ℓ is also defined, where $\Phi(\ell)$ means $\sum_{i=1}^{n} \varphi(k_i) \bmod 2$.

For two links $\ell(\subset R^3[0]), \ell'(\subset R^3[1])$, ℓ and ℓ' are said to be *related* if there is a locally flat non-singular orientable surface F of genus 0 in $R^3[0,1]$ with $F \cap R^3[0] = \ell$ and $F \cap R^3[1] = -\ell'$.

Lemma 2. 1([4]). *Two links ℓ and ℓ' are proper and related, then $\varphi(\ell) = \varphi(\ell')$*

For n-component links L, L', suppose that L and L' are homotopic, namely there is a homotopy h_t of $R^3 \times [0,1]$ such that $h_0(L) = L$ and $h_1(L) = L'$. For a sublink $\mathcal{L}$ of L, $\mathcal{L}'(=h_1(\mathcal{L}))$ is called the sublink of L' *corresponding to* $\mathcal{L}$ (with respect to h_t).

The following is valid to the classification by self $\sharp$-move and self pass-move of homotopic links.

Lemma 2. 2([6]) *Let L and L' be homotopic links. Then the following (1) and (2) hold.*

(1) L and L' are self $\sharp$-equivalent if and only if $\bar{\varphi}(\mathcal{L}) = \bar{\varphi}(\mathcal{L}')$ for any pair of sublinks $\mathcal{L}$ of L and $\mathcal{L}'$ of L' corresponding to $\mathcal{L}$ which are proper.

(2) L and L' are self pass-equivalent if and only if $\varphi(\mathcal{L}) = \varphi(\mathcal{L}')$ for any pair of sublinks $\mathcal{L}$ of L and $\mathcal{L}'$ of L' corresponding to $\mathcal{L}$ which are proper

Theorem 2.3. *Let $\mathcal{F} = F_1 \cup ... \cup F_n, \mathcal{F}' = F_1' \cup ... \cup F_n'$ be unions of mutually disjoint orientable surfaces of genus 0 with $\partial F_i = k_{i0} \cup ... \cup k_{im_i}$ and $\partial F_i' = k_{i0}' \cup ... \cup k_{im_i}'$ and let $\ell = \partial\mathcal{F}, L = k_{10} \cup ... \cup k_{n0}$ and $\ell' = \partial\mathcal{F}', L' = k_{10}' \cup ... \cup k_{n0}'$.*

Suppose that ℓ, ℓ' are homotopic and that L' is corresponding to L. If ℓ is purely proper and $\ell - L, \ell' - L'$ are self $\natural$-equivalent, then ℓ and ℓ' are self $\natural$-equivalent. Moreover if $\varphi(k_{ij}) = \varphi(k_{ij}')$ for each $i = 1, ..., n, j = 1, ..., m_i$, then ℓ and ℓ' are self pass-equivalent.

Proof. Since ℓ and ℓ' are homotopic and ℓ is purely proper, ℓ' is also purely proper and so, for any sublinks of ℓ, ℓ', the Arf invariants of these links are defined.

To prove that ℓ and ℓ', which are homotopic, are self $\natural$-equivalent, it is sufficient to do that, for any sublinks ℓ_r of ℓ with r-component and ℓ_r' of ℓ' corresponding to $\ell_r, \bar{\varphi}(\ell_r) = \bar{\varphi}(\ell_r')$ by Lemma 2.2(1).

If ℓ_r is contained in $\ell - L, \ell_r'$ is also contained in $\ell' - L'$ and so ℓ_r and ℓ_r' are self $\natural$-equivalent, because $\ell - L$ and $\ell' - L'$ are self $\natural$- equivalent. Hence $\bar{\varphi}(\ell_r) = \bar{\varphi}(\ell_r')$ by Lemma 2.2(1).

Therefore we assume that ℓ_r is not contained in $\ell - L$, namely there are some distinct integers $i_1, ..., i_p$ such that $\ell_r = k_{i_1 0} \cup ... \cup k_{i_p 0} \cup (\ell_r - L)$. To simplify the proof, we may assume that $i_j = j$ for $j = 1, ..., p$. Let us denote $\ell_r \cap (\bigcup_{i=1}^{p} \partial F_i), \ell_r \cap (\bigcup_{i=p+1}^{n} \partial F_i)$ and $(\ell - \ell_r) \cap (\bigcup_{i=1}^{p} \partial F_i)$ by $\ell_{r,0}, \ell_{r,1}$ and $\mathcal{L}$ respectively. Since $g(\mathcal{F})$, genus of $\mathcal{F}$, is 0, $\ell_r (= \ell_{r,0} \cup \ell_{r,1})$ and $\mathcal{L} \cup \ell_{r,1} (\subset \ell - L)$ are related and proper, $\varphi(\ell_r) = \varphi(\mathcal{L} \cup \ell_{r,1})$ by Lemma 2.1.

Therefore, for a link ℓ_r of ℓ,

$$\bar{\varphi}(\ell_r) \equiv \varphi(\ell_r) - \Phi(\ell_r) \equiv \varphi(\mathcal{L} \cup \ell_{r,1}) - \Phi(\ell_r) \equiv \bar{\varphi}(\mathcal{L} \cup \ell_{r,1}) + \Phi(\mathcal{L} \cup \ell_{r,1}) - \Phi(\ell_r) \equiv \bar{\varphi}(\mathcal{L} \cup \ell_{r,1}) + \Phi(\mathcal{L}) + \Phi(\ell_{r,0}) \equiv \bar{\varphi}(\mathcal{L} \cup \ell_{r,1}) + \Phi(\textstyle\bigcup_{i=1}^{p} \partial F_i) \pmod 2.$$

As $g(F_i) = 0, k_{i0}$ is related to $\partial F_i - k_{i0}$ and so $\varphi(k_{i0}) = \varphi(\partial F_i - k_{i0})$ by Lemma 2.1. Hence,

$$\Phi(\partial F_i) \equiv \varphi(k_{i0}) + \Phi(\partial F_i - k_{i0}) \equiv \varphi(\partial F_i - k_{i0}) + \Phi(\partial F_i - k_{i0}) \equiv \bar{\varphi}(\partial F_i - k_{i0}) \pmod 2.$$

Therefore we obtain that

$$\bar{\varphi}(\ell_r) \equiv \bar{\varphi}(\mathcal{L} \cup \ell_{r,1}) + \textstyle\sum_{i=1}^{p} \bar{\varphi}(\partial F_i - k_{i0}) \pmod 2.$$

By the same discussion as above, we obtain that

$$\bar{\varphi}(\ell_r') \equiv \bar{\varphi}(\mathcal{L}' \cup \ell_{r,1}') + \textstyle\sum_{i=1}^{p} \bar{\varphi}(\partial F_i' - k_{i0}') \pmod 2,$$

where $\mathcal{L}'$ and $\ell_{r,1}'$ are the sublinks of ℓ' corresponding to $\mathcal{L}$ and $\ell_{r,1}$ respectively. Since both $\mathcal{L} \cup \ell_{r,1}$ and $\partial F_i - k_{i0}$ are contained in ℓ-L and both $\mathcal{L}' \cup \ell_{r,1}'$ and $\partial F_i' - k_{i0}'$ are contained in $\ell' - L', \mathcal{L} \cup \ell_{r,1}$ and $\partial F_i - k_{i0}$ are self $\natural$-equivalent to $\mathcal{L}' \cup \ell'_{r,1}$ and $\partial F'_i - k'_{i0}$ respectively. Hence $\bar{\varphi}(\mathcal{L} \cup \ell_{r,1}) = \bar{\varphi}(\mathcal{L}' \cup \ell'_{r,1})$ and $\bar{\varphi}(\partial F_i - k_{i0}) = \bar{\varphi}(\partial F'_i - k'_{i0})$ for each i by Lemma 2.2(1) and so we obtain that $\bar{\varphi}(\ell_r) = \bar{\varphi}(\ell'_r)$. Therefore ℓ and ℓ' are self $\sharp$ -equivalent by Lemma 2.2(1).

Moreover if $\varphi(k_{ij}) = \varphi(k'_{ij})$ for each $i = 1, ..., n, j = 1, ..., m_i$, then $\Phi(\partial F_i - k_{i0}) = \Phi(\partial F'_i - k'_{i0})$. Furthermore as ℓ and ℓ' are self $\sharp$-equivalent,

$\varphi(k_{i0}) = \varphi(\partial F_i - k_{i0}) \equiv \bar{\varphi}(\partial F_i - k_{i0}) + \Phi(\partial F_i - k_{i0}) \equiv \bar{\varphi}(\partial F'_i - k'_{i0}) + \Phi(\partial F'_i - k'_{i0}) \equiv \varphi(k'_{i0}) \pmod 2$.

Therefore we obtain that $\varphi(\ell_r) = \varphi(\ell'_r)$ and so ℓ and ℓ' are self pass-equivalent by Lemma 2.2(2).

Remark 2.4. In Theorem 2.3, the condition "purely proper" is essential. For example, although the links illustrated in Fig. 6 in [5] satisfy the other conditions except "purely proper" of Theorem 2.3, they are not self $\sharp$-equivalent.

Theorem 2.5. *Let $\ell = k_1 \cup ... \cup k_n, \ell' = k'_1 \cup ... \cup k'_n$ be n -component links of genus 0 which are homotopic.*

If $\Phi(\ell) = \Phi(\ell')$ and, for any $(n-2)$ -component sublinks ℓ_{n-2} of ℓ and ℓ'_{n-2} of ℓ' corresponding to ℓ_{n-2}, ℓ_{n-2} and ℓ'_{n-2} are self $\sharp$ -equivalent, then ℓ and ℓ' are self $\sharp$-equivalent.

Moreover if $\varphi(k_i) = \varphi(k'_i)$ for each $i = 1, ..., n$, then ℓ and ℓ' are self pass-equivalent.

Proof. Since $g(\ell) = 0, \ell$ is related to a trivial knot. Hence if ℓ is proper, $\varphi(\ell) = 0$ by Lemma 2.1 and so $\bar{\varphi}(\ell) = \Phi(\ell)$. For an $(n-1)$-component link ℓ_{n-1}, say for example $\ell_{n-1} = k_2 \cup ... \cup k_n, \ell_{n-1}$ is related to k_1. Hence if ℓ_{n-1} is proper, $\varphi(\ell_{n-1}) = \varphi(k_1)$ and so $\bar{\varphi}(\ell_{n-1}) = \Phi(\ell)$. If ℓ (or ℓ_{n-1}) is proper, ℓ' (resp. ℓ'_{n-1}) is also proper and the reduced Arf invariant of ℓ (resp. ℓ_{n-1}) coincides with that of ℓ' (resp. ℓ'_{n-1}) by the assumption. Hence we obtain that ℓ and ℓ' are self $\sharp$ -equivalent by Lemma 2.2(1).

Moreover if $\varphi(k_i) = \varphi(k'_i)$ for each i, we obtain that $\varphi(\ell_r) = \varphi(\ell'_r)$ for any sublinks ℓ_r of ℓ and ℓ'_r of ℓ' corresponding to ℓ_r which are proper. Hence ℓ and ℓ' are self pass-equivalent by Lemma 2.2(2).

Corollary 2.6. *Let ℓ and ℓ' be those of Theorem 2.5.*

(1) *Let ℓ be a 2- or 3-component link. Then ℓ and ℓ' are self $\sharp$-equivalent if and only if $\Phi(\ell) = \Phi(\ell')$.*

(2) *Let ℓ be a 4-component link. Suppose that, for 2-component sublinks ℓ_2 of ℓ and ℓ'_2 of ℓ' corresponding to ℓ_2,*

(*a*) *each ℓ_2 is not proper (i.e., the linking number of each 2-component link of ℓ is odd) or that*

(*b*) *$\bar{\varphi}(\ell_2) = \bar{\varphi}(\ell'_2)$ for each ℓ_2 which is proper.*

Then ℓ and ℓ' are self $\sharp$ -equivalent if and only if $\Phi(\ell) = \Phi(\ell')$.

Proof. (1) First we consider the case that ℓ is 2-component. Since $g(\ell) = g(\ell') = 0, k_1 \approx k_2$ and $k_1' \approx k_2'$ and so $\Phi(\ell) = \Phi(\ell') = 0$.

The converse is easily obtained by Theorem 2.5.

Next we consider the case that ℓ is 3-component. If ℓ is proper, ℓ' is also proper and $\varphi(\ell) = \varphi(\ell') = 0$. Hence if ℓ and ℓ' are self $\sharp$-equivalent, $\Phi(\ell) = \bar{\varphi}(\ell) = \bar{\varphi}(\ell') = \Phi(\ell')$. If ℓ is not proper, one of $Link(k_i, \ell - k_i), 1 \leq i \leq 3$, for example $Link(k_1, \ell - k_1)$, is odd. So one of $Link(k_1, k_2)$ or $Link(k_1, k_3)$ say $Link(k_1, k_2)$, is even. Then $\ell_2 = k_1 \cup k_2$ is proper and related to k_3. Hence $\varphi(\ell_2) = \varphi(k_3)$ and so $\bar{\varphi}(\ell_2) = \Phi(\ell)$.

For a sublink $\ell_2' = k_1' \cup k_2'$ of ℓ' corresponding to ℓ_2, we also obtain that $\varphi(\ell_2') = \varphi(k_3')$ and $\bar{\varphi}(\ell_2') = \Phi(\ell')$.

If ℓ and ℓ' are self $\sharp$- equivalent, $\ell_2$and ℓ_2' are also self $\sharp$-equivalent and hence we obtain that $\Phi(\ell) = \bar{\varphi}(\ell_2) = \bar{\varphi}(\ell_2') = \Phi(\ell')$.

Since a $\sharp$-move is an unknotting operation of knots,[2], the converse is obtained by Theorem 2.5.

(2) Secondly we consider the case that ℓ is 4-component.

If $\Phi(\ell) = \Phi(\ell'), \ell$ and ℓ' are self $\sharp$-equivalent by Lemma 2.2 and Theorem 2.5.

Next suppose that ℓ and ℓ' are self $\sharp$-equivalent.

First we consider the case (a). As each ℓ_2 is not proper, any 3-component link ℓ_3 of ℓ is proper and ℓ_3' of ℓ' corresponding to ℓ_3 is also proper. Since ℓ_3 and ℓ_3' are self $\sharp$-equivalent and $g(\ell) = g(\ell') = 0$, $\Phi(\ell) = \bar{\varphi}(\ell_3) = \bar{\varphi}(\ell_3') = \Phi(\ell')$ by Lemma 2.2(1) and the proof of Theorem 2.5.

Secondly we consider the case (b). If ℓ is proper, then $\varphi(\ell) = \varphi(\ell') = 0$ and as $\ell. \ell'$ are self $\sharp$-equivalent, we obtain that $\Phi(\ell) = \bar{\varphi}(\ell) = \bar{\varphi}(\ell') = \Phi(\ell')$. If there is a 3-component sublink ℓ_3 of ℓ which is proper, then a link ℓ_3' of ℓ' corresponding to ℓ_3 is also proper and as they are self $\sharp$-equivalent, we obtain that $\Phi(\ell) = \Phi(\ell')$ by the discussion of (a). Therefore we assume that ℓ and any 3-component sublink $\mathcal{L}_i (= \ell - k_i, i = 1, 2.3, 4)$ of ℓ are not proper. Since ℓ is not proper, there is an integer i, for example $i = 1$, such that $Link(k_1, \ell - k_1)$ is odd and so one of $Link(k_1, k_j)$ is odd for $j = 2, 3, 4$, for example $Link(k_1, k_2)$ is odd. Now we consider the following 2 steps.

Step 1. $Link(k_1,\ k_3)$is odd.

Since $Link(k_1, \ell - k_1)$ is odd, $Link(k_1, k_4)$ is also odd. Hence as $\mathcal{L}_2, \mathcal{L}_3$, and $\mathcal{L}_4$ are not proper, $Link(k_2, k_3)$, $Link(k_2, k_4)$and $Link(k_3, k_4)$ are even. Then $\mathcal{L}_1$ is proper which contradicts to that $\mathcal{L}_1$ is not proper.

Step 2. $Link(k_1, k_3)$ is even.

Since $Link(k_1, \ell - k_1)$ is odd, $Link(k_1, k_4)$ is even. Hence as $\mathcal{L}_2$ is not proper, $Link(k_3, k_4)$ is odd and so as $\mathcal{L}_1$ is not proper, $Link(k_2, k_3)$ or $Link(k_2, k_4)$ is even, for example $Link(k_2, k_4)$ is even. Therefore both $k_1 \cup k_3$ and $k_2 \cup k_4$ are proper and related, $\varphi(k_1 \cup k_3) = \varphi(k_2 \cup k_4)$ and so

$\bar{\varphi}(k_1 \cup k_3) \equiv \varphi(k_1 \cup k_3) - \varphi(k_1) - \varphi(k_3) \equiv \varphi(k_2 \cup k_4) - \varphi(k_1) - \varphi(k_3) \equiv \bar{\varphi}(k_2 \cup k_4) - \Phi(\ell) \pmod 2$.

By applying the same discussion to ℓ', we obtain that

$\bar{\varphi}(k_1' \cup k_3') \equiv \bar{\varphi}(k_2' \cup k_4') - \Phi(\ell') \pmod 2$.

Since $k_1 \cup k_3$ and $k_2 \cup k_4$ are self $\sharp$-equivalent to $k_1' \cup k_3'$ and $k_2' \cup k_4'$ respectively, we obtain that $\Phi(\ell) = \Phi(\ell')$ by Lemma 2.2(1).

3 Special case: Links as the boundaries of mutually disjoint annuli.

In this section, we consider the self $\sharp$- and self pass -equivalence for links of genus 0 as the boundaries of m utually disjoint annuli. Namely let $\mathcal{A} = A_1 \cup \ldots \cup A_n$ and $\mathcal{A}' = A_1' \cup \ldots \cup A_n'$ be unions of mutually disjoint annuli A_i, A_i' with $\partial A_i = k_i \cup K_i, \partial A_i' = k_i' \cup K_i'$ for $i = 1, \ldots, n$. Denote $\partial\mathcal{A}, \partial\mathcal{A}'$ by ℓ, ℓ' and $\bigcup_{i=1}^n k_i, \bigcup_{i=1}^n k_i'$ by L, L' respectively.

Theorem 3.1. *With the above notation, assume that ℓ and ℓ' are homotopic. Then ℓ and ℓ' are self $\sharp$ (or self pass)-equivalent if and only if L and L' are self $\sharp$ (or self pass) -equivalent.*

Proof. The necessity is obvious. To prove the sufficiency, it is enough to do that, for any sublinks ℓ_r of ℓ and ℓ_r' of ℓ' corresponding to ℓ_r which are proper, $\bar{\varphi}(\ell_r) = \bar{\varphi}(\ell_r')$ (resp. $\varphi(\ell_r) = \varphi(\ell_r'$.))by Lemma 2.2(1) (resp. 2.2(2)). The above is obtained by the same discussion to that of proof of Lemma 4 in [5].

Remark 3.2. Under the conditions of Theorem 3.1, it is unnecessary that the condition such that ℓ is purely proper in Theorem 2.3.

The definition of self Δ-equivalence, see [3],[5],[6].

Corollary 3.3. *Let ℓ ℓ' L and L' be those of Theorem 3.1. If L and L' are self Δ-equivalent, then ℓ and ℓ' are self $\sharp$-equivalent.*

Proof. Since L and L' are self Δ-equivalent, we obtain that ℓ and ℓ' are quasi self Δ-equivalent by the deformations in Fig. 3 in [5] and hence ℓ and ℓ' are homotopic by [3]. Moreover if L and L' are self Δ -equivalent, we easily see that, for any sublink $\mathcal{L}$ of L and that of L' corresponding to $\mathcal{L}$ which are proper, their reduced Arf invariants coincide and so L and L' are self $\sharp$-equivalent.

Therefore ℓ and ℓ' are self $\natural$ -equivalent by Theorem 3.1.

Corollary 3.3 is an extention of Lemma 4 in [5].

It is well-known that two 2-component links are homotopic if and only if their linking numbers coincide, [1]. Therefore we obtain the following by Lemma 2.2(1) and Theorem 3.1.

Corollary 3.4. *Let* $n=2$ *in Theorem* 3.1 *and* $s = Link(k_1, k_2)$
(1) *If* s *is odd,* ℓ, ℓ' *are self* $\natural$*-equivalent.*
(2) *If* s *is even, then* ℓ *and* ℓ' *are* $\natural$*-equivalent if and only if* $\bar{\varphi}(L) = \bar{\varphi}(L')$.

References

[1] J.W. Milnor: Link groups, Ann. of Math. **59**(1954), 177-195.
[2] H. Murakami: Some metrics on classical knots, Math. Ann. **270**(1985),35-45.
[3] Y. Nakanishi and T. Shibuya: Link homotopy and quasi self delta-equivalence for links, J. of Knot Theory and its Ramification, **9**(2000), 683-691.
[4] R.A. Robertello: An invariant of knot cobordism, Comm. Pure and Appl. Math. **18**(1965), 543-555.
[5] T. Shibuya: On local moves for links of genus zero, Memo. Osaka Insti. Tech., **47**(2002), 31-39.
[6] T. Shibuya and A. Yasuhara: Classification of links up to self pass-move, J. Math. Soc. Japan, **55**(2003), to appear.