Memoirs of the Osaka Institute
of Technology, Series A
Vol.49,No.1(2004) pp.11～19

FUSION AND SELF SHARP EQUIVALENCE OF LINKS

by

Tetsuya HIROHATA Tetsuo SHIBUYA

Department of General Education, Faculty of Engineering
(Manuscript received May 26, 2004)

Abstract

J.W.Milnor introduced a local move called the link-homotopy and it is an important tool for the classification of links.

Recently several local moves of links whose concepts are stronger than that of link-homotopy are defined. And the properties of these moves are studied and applied to the classification of links.

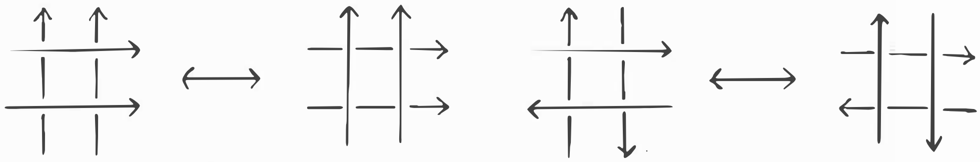
In this paper, we discuss a local move called the #-move and prove some propertieso f it and apply them to the classification of links obtained by fusions.

1 Introduction.

Throughout this paper, link is tame, oriented and ordered one in an oriented 3-space R^3.

For a link ℓ, the deformation illustrated in Fig. 1(a) ((b)) applied to a component of ℓ is called *a self* $\sharp$ (resp. *self pass*)*-move*. (In [4],[7],[8], these moves were called self $\sharp$(I)-move, self $\sharp$(II) -move respectively.)

Two links ℓ and ℓ' are said to be *self* $\sharp$ (or *self pass*)*-equivalent* or ℓ is said to be *self* $\sharp$ (resp. *self pass*)*-equivalent to* ℓ' if ℓ can be deformed into ℓ' by a finite sequence of self $\sharp$ (resp. self pass)-moves.



(a) (b)

Fig. 1

If ℓ and ℓ' are self pass-equivalent, they are self $\sharp$-equivalent [3]. But the converse is not true. The following is known [4], where $\varphi(k)$ means the Arf invariant of a knot k [5].

Proposition. *For two n-component links* $\ell = k_1 \cup ... \cup k_n$ *and* $\ell' = k'_1 \cup ... \cup k'_n$*, if* ℓ *and* ℓ' *are self* $\sharp$*-equivalent and* $\varphi(k_i) = \varphi(k'_i)$ *for each* $i = 1, ..., n$*, then* ℓ *and* ℓ' *are self pass-equivalent.*

Although we consider some properties of self $\sharp$-equivalence of links only , the obtained results in this paper may apply to self pass-equivalence of links by adding the condition of the Arf invariant of knots.

Let $L = K_1 \cup ... \cup K_n$ be an n-component link. Suppose that $\mathcal{B} = B_1 \cup ... \cup B_p$ is a union of mutually disjoint disks B_i for $p < n$ such that $B_i \cap L = \partial B_i \cap L$ which consists of two arcs of L with orientation coherently to that of L for each $i = 1, ..., p$ and $\tilde{L}(= L + \partial\mathcal{B})$ is an $(n-p)$-component link. Then we say that $\tilde{L}$ *is obtained by a p-fusion of* L and that $\mathcal{B}$ are *disks of a p-fusion of* L.

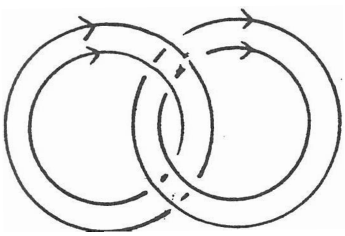
In Section 2, we study the self $\sharp$-equivalence of links obtained by p-fusions of self $\sharp$-equivalent links and prove Theorems 2.2 and 2.4.

Next we consider a special p-fusion, called *a product p-fusion*. If $L' = K_1' \cup ... \cup K_n'$ is an n- component link *split* from the above L in R^3, namely there is a 3-ball E^3 in R^3 such that $L \subset E^3$ and $L' \cap E^3 = \emptyset$, then we denote $L \cup L'$ by $L \circ L'$. For a $2n$-component link $L \circ L'$, let $\mathcal{B} = B_1 \cup ... \cup B_n$ be disks of an n-fusion of $L \circ L'$ such that $B_i \cap (L \circ L')$=(an arc of K_i)$\cup$(an arc of K_i') for each B_i, $i = 1, ..., n$. Then $\tilde{L}(= L \circ L' + \partial \mathcal{B})$ is called *a link obtained by a product n fusion of $L \circ L'$*.

In Section 3, we study the self $\sharp$-equivalence of links obtained by product n-fusions of split links and prove Theorems 3.1 and 3.3.

2 Fusions and self $\sharp$-move of links.

For two n-component links $\ell = k_1 \cup ... \cup k_n, \ell' = k_1' \cup ... \cup k_n'$ in $R^3[a], R^3[b]$ respectively, if there is a union of mutually disjoint annuli $\mathcal{A} = A_1 \cup ... \cup A_n$ in $R^3[a, b]$ satisfying the following, we say that $\mathcal{A}$ is the union of *$\sharp$-annuli between* ℓ and ℓ': $\mathcal{A} \cap R^3[a] = \ell$, $\mathcal{A} \cap R^3[b] = -\ell'$ and $A_i \cap R^3[a] = k_i, A_i \cap R^3[b] = -k_i'$ for each annulus A_i and $\mathcal{A}$ is locally flat and non-singular except finite points in the interior of $\mathcal{A}$ which are the singularities of $\mathcal{A}$, denoted by $\mathcal{S}(\mathcal{A})$, such that $(\partial N(P: R^3[a, b]), \partial N(P: \mathcal{A}))$ is the link illustrated in Fig. 2 for each point P of $\mathcal{S}(\mathcal{A})$. In this case, ℓ and ℓ' are said to be *$\sharp$-cobordant* or ℓ is said to be *$\sharp$-cobordant to ℓ'*.



or

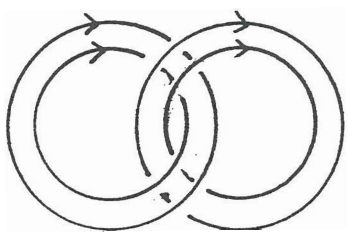


Fig. 2

We easily see that, if ℓ and ℓ' are self $\sharp$-equivalent, they are level-preserving $\sharp$-annuli $\mathcal{A}$, namely $\mathcal{A}$ has neither minimal nor maximal points [2], between ℓ and ℓ'. Moreover we obtain the following by the similar way to the proof of Lemma 1. 17 in [6].

Lemma 2.1. *For two links ℓ and ℓ' in R^3, ℓ and ℓ' are self $\sharp$-equivalent if and only if they are $\sharp$-cobordant.*

Theorem 2.2. *Let ℓ and ℓ' be n-component links which are self $\sharp$-equivalent. For any integer $p(1 \leq p < n)$, let L be a link obtained by any p fusion of ℓ. Then there is a link L' which is obtained by a p fusion of ℓ' such that L and L' are self $\sharp$-equivalent.*

Proof. As $\ell(\subset R^3[0])$ and $\ell'(\subset R^3[2])$ are self $\sharp$-equivalent, there is a union of mutually disjoint level-preserving $\sharp$-annuli $\mathcal{A}(= A_1 \cup ... \cup A_n)$ in $R^3[0,2]$ between ℓ and ℓ'. Let $P_1, ..., P_q$ be the points of $\mathcal{S}(\mathcal{A})$ and $\alpha_1, ..., \alpha_q$ mutually disjoint level-preserving arcs in $R^3[-1,2]$ such that $\alpha_i \cap \mathcal{A} = P_i$ and $\partial\alpha_i = P_i \cup Q_i$ for a point Q_i in $R^3[-1]$. By deforming $\mathcal{A}$ along α_i from P_i to Q_i with $\partial\mathcal{A}$ fixed, we obtain annuli $\mathcal{A}'$ in $R^3[-1,2]$ with $\partial\mathcal{A}' = \partial\mathcal{A}$. Let $\mathcal{F} = \mathcal{A}' \cap R^3[0,2]$. Then $\mathcal{F}$ is a locally flat non-singular orientable surface of genus 0 with $\mathcal{F} \cap R^3[0] = \ell \circ \mathcal{L}$ and $\mathcal{F} \cap R^3[2] = -\ell'$, where $\mathcal{L}$ consists of q links illustrated in Fig. 2. By an isotopy of $R^3[0,2]$ with $\mathcal{F} \cap R^3[0]$ fixed, we obtain a surface $\mathcal{F}_0$ from $\mathcal{F}$ such that $\partial\mathcal{F}_0 \cap R^3[0] = \ell \circ \mathcal{L}$ and $\partial\mathcal{F}_0 \cap R^3[2] \approx -\ell'$ and there are disks $\mathcal{C}$ in $R^3[1]$ of a $4q$-fusion of $(\ell \circ \mathcal{L}) \times \{1\}$ with $\mathbf{L} = ((\ell \circ \mathcal{L}) \times \{1\}) + \partial\mathcal{C} \approx \ell'$ satisfying that $\mathcal{F}_0 - \mathcal{C} \times [0,2] = (\mathcal{F}_0 \cap R^3[1] - \mathcal{C}) \times [0,2]$, where $X \times \{i\}$ means the projection of X into $R^3[i]$.

Since ℓ is split from $\mathcal{L}$, there is a 3-ball E^3 in $R^3[0]$ such that $E^3 \cap (\ell \circ \mathcal{L}) = \mathcal{L}$. As L is obtained by a p-fusion of ℓ, there are mutually disjoint disks $\mathcal{B}$ in $R^3[0]$ such that $L = \ell + \partial\mathcal{B}$. Furthermore we may choose $\mathcal{B}$ such that $\mathcal{B} \cap (E^3 \cup \mathcal{C} \times \{0\}) = \emptyset$ and so $\mathcal{F}_0 \cap (\mathcal{B} \times [0,2]) = \mathcal{F}_0 \cap (\partial\mathcal{B} \times [0,2])$ by the construction of $\mathcal{F}_0$. Then $F = cl((\mathcal{F}_0 - (\partial\mathcal{B} \cap \ell) \times [0,2]) \cup (\partial\mathcal{B} - \ell) \times [0,2])$ is a non -singular orientable surface of genus 0 in $R^3[0,2]$ with $\partial F \cap R^3[0] = L \circ \mathcal{L}$ by the choice of $\mathcal{B}$. Since $\mathcal{L}$ consists of q links illustrated in Fig. 2 which is split from L, we may construct a union $\mathcal{A}_0$ of mutually disjoint $\sharp$-annuli in $R^3[0,2]$ by using F between L and $\partial\mathcal{A}_0 \cap R^3[2] (= \partial F \cap R^3[2])$ which is ambient isotopic to a link obtained by a p-fusion of ℓ'. Hence we obtain Theorem 2.2 by Lemma 2.1.

For a link $\ell = k_1 \cup ... \cup k_n$, ℓ is said to be *proper* if the linking number $Link(k_i, \ell - k_i)(= \sum_{j \neq i} Link(k_i, k_j))$ is even for each $i = 1, ..., n$. The Arf invariant $\varphi(\ell)$ is defined if ℓ is proper [5]. Hence the reduced Arf invariant, denoted by $\bar{\varphi}(\ell)(\equiv \varphi(\ell) - \sum_{i=1}^{n} \varphi(k_i) \bmod 2)$, is also defined if ℓ is proper.

For the self $\sharp$-equivalence of 2-component links, the following is known in [8]. (Recently it is proved that the self $\sharp$-equivalence of homotopic links can be classified by these reduced Arf invariants of proper sublinks [9].)

Lemma 2.3. *Let $\ell = k_1 \cup k_2, \ell' = k_1' \cup k_2'$ be 2-component links respectively with $Link(k_1, k_2) = Link(k_1', k_2')(= r)$. Then*

(1) *If r is odd, ℓ and ℓ' are self $\sharp$-equivalent.*

(2) *If r is even, ℓ and ℓ' are self $\sharp$-equivalent if and only if $\bar{\varphi}(\ell) = \bar{\varphi}(\ell')$.*

For $p = n - 2$ in Theorem 2. 2, we obtain the following.

Theorem 2.4. *Suppose that $\ell = \ell_1 \cup \ell_2$ and $\ell' = \ell_1' \cup \ell_2'$ are n-component links which are self $\sharp$-equivalent and that $L = K_1 \cup K_2$ and $L' = K_1' \cup K_2'$ are links obtained by any $(n-2)$ fusions of ℓ and ℓ' respectively, where K_i and K_i' are knots obtained by $(n_i - 1)$-fusions of ℓ_i and ℓ_i' respectively for $i = 1, 2$ and $n_1 + n_2 = n$. Let $r = Link(\ell_1, \ell_2)(= \sum_{k_i \subset \ell_1, k_j \subset \ell_2} Link(k_i, k_j))$. Then*

(1) *If r is odd, L and L' are self $\sharp$-equivalent.*

(2) *If r is even and ℓ, ℓ_1 and ℓ_2 are proper, L and L' are self $\sharp$-equivalent.*

Proof. Since ℓ and ℓ' are self $\sharp$-equivalent, each ℓ_i and ℓ_i' is self $\sharp$-equivalent for $i = 1, 2$ and so $Link(\ell_1', \ell_2') = r$. Moreover, since both K_i and K_i' are obtained by $(n_i - 1)$-fusions of ℓ_i and ℓ_i' respectively, we obtain that $Link(K_1, K_2) = Link(K_1', K_2') = r$. Hence if r is odd, L and L' are self $\sharp$-equivalent by Lemma 2.3(1).

Next we consider the case when r is even. Then both L and L' are proper. Hence it is sufficient to prove that $\bar{\varphi}(L) = \bar{\varphi}(L')$ in this case by Lemma 2. 3(2).

Since ℓ and ℓ' are self $\sharp$-equivalent and ℓ_1, ℓ_2 are proper, ℓ_1', ℓ_2' are also proper and self $\sharp$-equivalent to ℓ_1, ℓ_2 respectively. Therefore we have $\varphi(\ell) = \varphi(L), \varphi(\ell') = \varphi(L'), \varphi(\ell_i) = \varphi(K_i), \varphi(\ell_i') = \varphi(K_i'), \bar{\varphi}(\ell) = \bar{\varphi}(\ell')$ and $\bar{\varphi}(\ell_i) = \bar{\varphi}(\ell_i')$ for $i = 1, 2$. Hence,

$$\begin{aligned}\bar{\varphi}(L) &\equiv \varphi(L) - \varphi(K_1) - \varphi(K_2) \equiv \varphi(\ell) - \varphi(\ell_1) - \varphi(\ell_2) \\ &\equiv \bar{\varphi}(\ell) - \bar{\varphi}(\ell_1) - \bar{\varphi}(\ell_2) \quad \equiv \bar{\varphi}(\ell') - \bar{\varphi}(\ell_1') - \bar{\varphi}(\ell_2') \equiv \bar{\varphi}(L') \pmod 2.\end{aligned}$$

Therefore L and L' are self $\sharp$-equivalent.

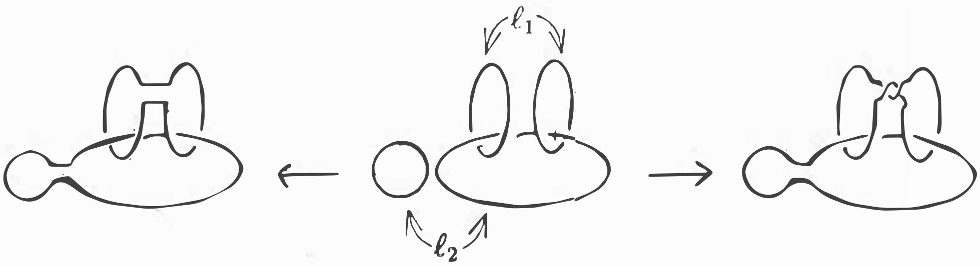
For an n-component link $\ell = k_1 \cup ... \cup k_n$, ℓ is said to be *purely proper* if $Link(k_i, k_j)$ is even for each $i, j = 1, .., n, i \neq j$. If ℓ is purely proper, ℓ and any sublink of ℓ are proper. By Theorem 2.4 we obtain

Corollary 2.5. *Let ℓ, ℓ' and L, L' be those of Theorem 2.4. If ℓ is purely proper, L and L' are self $\sharp$-equivalent.*

Remark 2.6. In Theorem 2.4(2), if one of ℓ, ℓ_1 or ℓ_2 is not proper, there are links L and L' which are not self $\sharp$-equivalent.

Example 1. *ℓ is not proper.*

Two links L and L' obtained by 2-fusions of a link ℓ in Fig. 3 are not self $\sharp$-equivalent by Lemma 2.3, because $\bar{\varphi}(L) = 0$ and $\bar{\varphi}(L') = 1$.

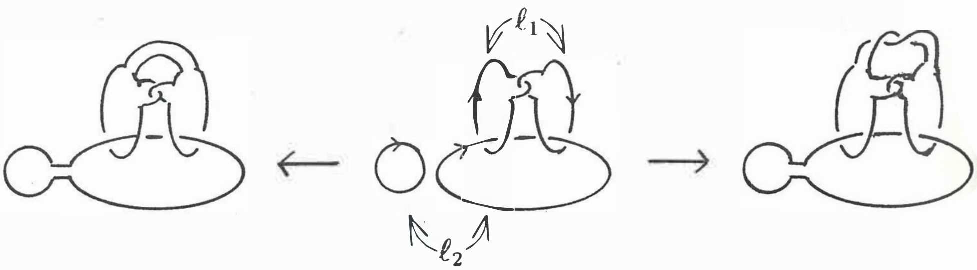


L ℓ L'

Fig. 3

Example 2. *ℓ_1 is not proper.*

Two links L and L' obtained by 2-fusions of ℓ in Fig. 4 are not self $\sharp$-equivalent, because $\bar{\varphi}(L) = 0$ and $\bar{\varphi}(L') = 1$.



L ℓ L'

Fig. 4

3 Product n-fusions and self $\sharp$-move of links.

In this section, we consider the self $\sharp$-equivalence of links obtained by product fusions.

Theorem 3.1. *Let ℓ and ℓ' be n-component links. Then ℓ and ℓ' are self $\sharp$-equivalent if and only if there is a product n-fusion of $\ell \circ (-\ell')$such that the link L_0 obtained by the fusion is self $\sharp$-equivalent to a trivial link.*

Proof. Suppose that $\ell(\subset R^3[0])$ and $\ell'(\subset R^3[2])$ are self $\sharp$-equivalent. Then there are a surface $\mathcal{F}_0(\subset R^3[0,2])$ and disks $\mathcal{C}(\subset R^3[1])$ having the same properties of the proof of Theorem 2.2.

It is known that there is a product n-fusion of $\ell \circ (-\ell)$such that the link L obtained by the fusion is a ribbon link [1]. Furthermore we may choose the disks $\mathcal{E}$ of the above product n-fusion of $\ell \circ (-\ell)$in $R^3[0]$ such that $\mathcal{E} \cap (E \cup \mathcal{C} \times \{0\}) = \emptyset$, where E^3 is that of the proof of Theorem 2.2 and $\mathcal{F}_0 \cap (\mathcal{E} \times [0,2]) = (\ell \cap \partial\mathcal{E}) \times [0,2]$ by the construction of $\mathcal{F}_0$. Therefore we may easily construct $\sharp$-annuli $\mathcal{A}_0$ in $R^3[0,2]$ between L and a link $L'(= \ell' \circ (-\ell) + \partial(\mathcal{E} \times \{2\}))$. Since L is a ribbon link, L is self $\sharp$-equivalent to a trivial link $\mathcal{O}$ [7]. Hence L' is $\sharp$-cobordant to $\mathcal{O}$ and so L' is self $\sharp$-equivalent to $\mathcal{O}$ by Lemma 2.1. Therefore we obtain the necessity.

Next let us prove the sufficiency. Suppose that $L', \mathcal{E}$ are those of the above. Then $L' + \partial(\mathcal{E} \times \{2\}) = \ell' \circ (-\ell)$and so we easily construct $\sharp$-annuli between ℓ' and ℓ. Hence we obtain that ℓ and ℓ' are self $\sharp$-equivalent by Lemma 2.1.

In general, the links obtained by product n-fusions of $\ell \circ \ell'$ are not unique up to self $\sharp$-equivalence. For example, let L, L' be two links obtained by product 2-fusions of 2 Hopf links, Fig. 5. Then L and L' are not self $\sharp$-equivalent by Lemma 2.3, because $\bar{\varphi}(L) = 0$ and $\bar{\varphi}(L') = 1$.

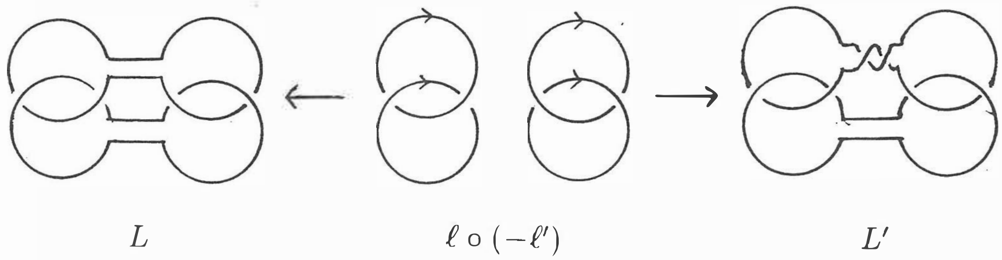


Fig. 5

But the following is true for 2-component links obtained by product 2-fusions.

Theorem 3.2. *Let $\ell = k_1 \cup k_2$ and $\ell' = k_1' \cup k_2'$ be 2-component links with $Link(k_1, k_2) = s$ and $Link(k_1', k_2') = s'$ and L, L' links obtained by product 2-fusions of $\ell \circ \ell'$. If s or s' is even, L and L' are self $\sharp$-equivalent.*

Proof. As both $L(= K_1 \cup K_2)$ and $L'(= K_1' \cup K_2')$ are obtained by product 2-fusions of $\ell \circ \ell'$, ℓ is split from ℓ' and so $Link(K_1, K_2) = Link(K_1', K_2') = s + s'$.

If s is even and s' is odd (or s is odd and s' is even), $s + s'$ is odd and so L and L' are self $\sharp$-equiva lentby Lemma 2.3(1).

Next suppose that both s and s' are even. Then $s + s'$ is even and so L, L' and $\ell \circ \ell'$ are proper and $\varphi(L) = \varphi(\ell \circ \ell') = \varphi(L')$ and $\varphi(K_i) = \varphi(k_i \circ k_i') = \varphi(K_i')$ by the construction of L, L', K_i and K_i' [5]. Therefore $\bar{\varphi}(L) = \bar{\varphi}(L')$ and hence L and L' are self $\sharp$-equivalent by Lemma 2.3.

By Theorems 3.1 and 3.2, we obtain the following.

Corollary 3.3. *Let $\ell(= k_1 \cup k_2)$ and ℓ' be 2-component links which are self $\sharp$-equivalent. If $Link(k_1, k_2)$ is even, any link obtained by product 2 fusion of $\ell \circ (-\ell')$ is self $\sharp$-equivalent to a trivial link.*

References

[1] F. Hosokawa: A concept of cobordism between links, Ann. of Math., **86**(1967), 362-373.

[2] A. Kawauchi, T. Shibuya and S. Suzuki: Descriptions on surfaces in four space, I, Math. Sem. Notes, Kobe Univ., **10**(1982), 75-125.

[3] H. Murakami and Y. Nakanishi: On a certain move generating link-homology, Math. Ann., **284**(1989), 75-88.

[4] Y. Nakanishi and T. Shibuya: Relations among self delta -equivalence and self sharp-equivalence for links, Knot in Hellas 98, C. Gordon(ed.), World Soc. Publ. Co. Ser. **24**, 353-360.

[5] R.A. Robeltello: An invariant of knot cobordism, Comm. Pure Appl. Math., **18**(1965), 543-555.

[6] T. Shibuya: On the cobordism of links in 3-space, Kobe J. Math., **1**(1984), 119-131.

[7] T. Shibuya: Self $\sharp$-unknotting operation of links, Memo. Osaka Insti. Tech., **34**(1989), 9-17.

[8] T. Shibuya: Mutation and self $\sharp$-equivalences of links, Kobe J. Math., **10**(1993), 23-37.

[9] T. Shibuya and A. Yasuhara: Classification of links up to self pass-move, J. Math. Soc. of Japan, **55**(2003), 939-946.