Memoirs of the Osaka Institute
of Technology, Series A
Vol. 54, No. 2(2009) pp. 1 ～ 8

Self Delta-Equivalence of Links Obtained by Product Fusions

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Abstract

It is known that two cobordant links can be transformed from one into the other by a finite sequence of link-homotopy (resp. self #-moves, self pass-moves). However it does not hold in the case of self Δ-moves. Let L be a link and L' the link obtained by certain product fusion of L and a separate ribbon link. It is easy to see that L and L' are cobordant. In this paper, we show that L and L' can be transformed from one into the other by a finite sequence of self Δ-moves.

Keywords; knots, links, self Δ-moves, fusions

1. Introduction

Throughout this paper, knots and links are ordered and oriented in a 3-space $\mathbb{R}^3$. A local move as illustrated in Figure 1 is called the Δ-*move* ([3], [4]). If the three strands in the figure belong to the same component of a link, we call the Δ-move the *self* Δ-*move*. We say that two links L and L' are *self* Δ-*equivalent* or that L is *self* Δ-*equivalent to* L' if L can be transformed into L' by a finite sequence of self Δ-moves and ambient isotopies.

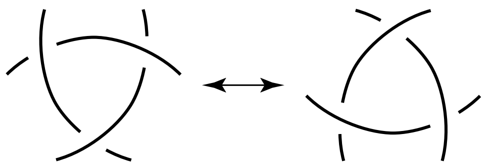

Figure 1

If a link $L \cup L'$ is *split* in $\mathbb{R}^3$, namely, there is a sphere S in $\mathbb{R}^3$ such that each connected component of $\mathbb{R}^3 - S$ contains either L or L', then we denote the link by $L \circ L'$ and we call S a *splitting sphere for* $L \circ L'$. Let $L = K_1 \cup \cdots \cup K_n$ and $L' = K'_1 \cup \cdots \cup K'_n$ be n-component links and let $\mathcal{B} = B_1 \cup \cdots \cup B_n$ be a disjoint union of bands for a fusion of L and L', where B_i is a band for a fusion of K_i and K'_i $(i = 1, \cdots, n)$. Then the link $(L \circ L') \oplus \partial\mathcal{B}$ is called a *link obtained by a fusion of* L *and* L', where $\oplus$ means the homological addition. Moreover if $B_i \cap S$ consists of an arc for each i, then $(L \circ L') \oplus \partial\mathcal{B}$ is called a *link obtained by a product fusion* of L *and* L', and denoted by $L \underset{\mathcal{B}}{\#} L'$ or simply by $L\#L'$.

It is known that there is a pair of links which are cobordant but not self Δ-equivalent ([6], [7]) (but if one of the pair is a trivial link, they are self Δ-equivalent [10]). A link $L = K_1 \cup \cdots \cup K_n$ is called a *separate ribbon link* if there is a disjoint union $\mathcal{D} = D_1 \cup \cdots \cup D_n$ of ribbon disks with $\partial\mathcal{D} = L$ and $\partial D_i = K_i$ $(i = 1, \cdots, n)$. Then the following is our main theorem.

Theorem 1.1. *Let ℓ be an n-component link and L an n-component separate ribbon link which bounds a disjoint union $\mathcal{D} = D_1 \cup \cdots \cup D_n$ of ribbon disks. Suppose that there is a sphere S such that each connected component of $\mathbb{R}^3 - S$ contains ℓ or $\mathcal{D}$. Let $\mathcal{B} = B_1 \cup \cdots \cup B_n$ be a disjoint union of bands for a product fusion of ℓ and L. If $B_i \cap D_j = \emptyset$ for each i and j $(i \neq j)$, then ℓ and $\ell\#L$ are self Δ-equivalent.*

For an n-component link ℓ and an n-component separate ribbon link L, a link obtained by a fusion of ℓ and L is cobordant to ℓ. However there is a link obtained by a fusion of ℓ and L which is not self Δ-equivalent to ℓ. For example, let H be a Hopf link, $\mathcal{O} = O_1 \cup O_2$ the 2-component trivial link, and $\mathcal{B}_1 = B_{11} \cup B_{12}$ the bands for a fusion of H and $\mathcal{O}$ as illustrated in the leftside of Figure 2. Then $(H \circ \mathcal{O}) \oplus \partial\mathcal{B}_1$ is cobordant to H, but not self Δ-equivalent to H ([6]). Thus Theorem 1.1 does not hold for a fusion (Note that this fusion is not a product fusion). In addition, the link $H \underset{\mathcal{B}_2}{\#} \mathcal{O}$ as illustrated in the rightside of Figure 2 is obtained by a product fusion and ambient isotopic to $(H \circ \mathcal{O}) \oplus \partial\mathcal{B}_1$. However since O_2 does not bound a non-singular disk without intersecting with B_{21}, condition "$B_i \cap D_j = \emptyset$ if $i \neq j$" is necessary in Theorem 1.1.

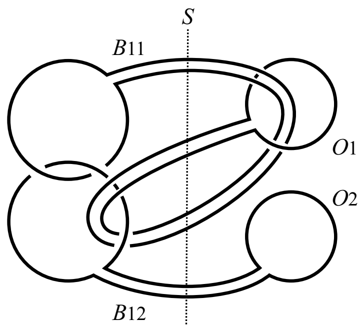

Figure 2

Next let $H\#M$ be the link obtained by a product fusion of the Hopf link H and a Milnor link M as illustrated in Figure 3. Although M is a ribbon link, it is known that M is not a separate ribbon link ([2]). Let $\mathcal{D} = D_1 \cup D_2$ be a union of ribbon disks as illustrated in Figure 3. Then we have that $\mathcal{D} \cap \mathcal{B} = \partial\mathcal{D} \cap \partial\mathcal{B}$. Since $a_3(H) = 0$, $a_3(H\#M) = 2$ for the third coefficient of the Conway polynomial, and any component of H (resp. $H\#M$) is trivial, $H\#M$ is not self Δ-equivalent to H from Theorem 3 in [5]. Hence condition "separateness" of L is necessary in Theorem 1.1.

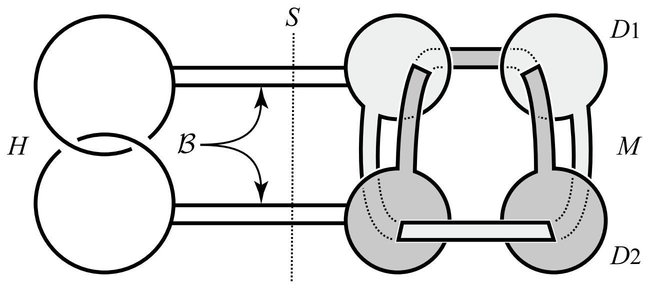

Figure 3

2. Proof of Theorem 1.1

To prove Theorem 1.1, we transform $\mathcal{B} \cup \mathcal{D}$ so that $\mathcal{D}$ is a disjoint union of non-singular disks and $\mathcal{S}(\mathcal{B} \cap \mathcal{D}) = \emptyset$, where $\mathcal{S}(\mathcal{B} \cap \mathcal{D})$ is the set of singularities of $\mathcal{B} \cap \mathcal{D}$. These transformations are realized either by isotopy or by self Δ-moves, which means that the links $(\ell \circ \partial\mathcal{D}) \oplus \partial\mathcal{B}$ of the before and the after of a transformation are isotopic or self Δ-equivalent. Before proving Theorem 1.1, we show the following special case that $\mathcal{D}$ is a disjoint union of non-singular disks.

Proposition 2.1. *Let ℓ be an n-component link and $\mathcal{O}$ the n-component trivial link which bounds a disjoint union $\mathcal{D} = D_1 \cup \cdots \cup D_n$ of non-singular disks. Suppose that there is a sphere S such that each connected component of $\mathbb{R}^3 - S$ contains ℓ or $\mathcal{D}$. Let $\mathcal{B} = B_1 \cup \cdots \cup B_n$ be a disjoint union of bands for a product fusion of ℓ and $\mathcal{O}$. If $B_i \cap D_j = \emptyset$ for each i and j $(i \neq j)$, then ℓ and $\ell\#\mathcal{O}$ are self Δ-equivalent.*

Each band B_i in int S is divided into open sub-bands by D_i $(i = 1, \cdots, n)$. Let U be the set of all such open sub-bands, that is, $U = \cup((B_i \cap \text{int } S) - D_i)$. Let b be an element of $U \cap B_i$. On the process of the proof, b intersects with D_j in ribbon arcs $(j < i)$. Let α and α' be singularities of $\mathcal{S}(b \cap D_j)$ and b' the open sub-band of b the ends of whose closure are α and α' (we say simply that α and α' are the *ends of* b). We call α and α' an *innermost intersection pair* if $\mathcal{S}(b' \cap \mathcal{D}) = \emptyset$, and $b' \cap (D_j \times [0,1]) = \emptyset$ or $b' \cap (D_j \times [0,-1]) = \emptyset$. We say that b is *well-situated with respect to* $\mathcal{D}_l = D_1 \cup \cdots \cup D_l$ if $\mathcal{S}(b \cap (\mathcal{D} - \mathcal{D}_l)) = \emptyset$ and we can reduce $\mathcal{S}(b \cap \mathcal{D}_l)$ to the empty set by removing innermost intersection pairs one by one.

Lemma 2.2. (*Lemma* 2.2, [9]) *The transformations as illustrated in Figure* 4 *are realized by* Δ*-moves.*

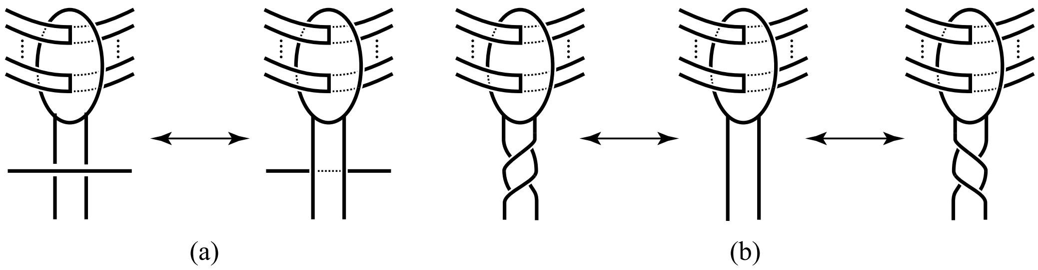

FIGURE 4

Proof of Proposition 2.1. Let $\ell = k_1 \cup \cdots \cup k_n$, $\ell \# \mathcal{O} = K_1 \cup \cdots \cup K_n$, where $K_i = (k_i \circ \partial D_i) \oplus \partial B_i$. We show that $\ell \# \mathcal{O}$ can be transformed into ℓ by applying self Δ-moves to $K_1, \cdots, K_n$ in this order. We transform $\mathcal{B} \cup \mathcal{D}$ so that $B_i \cap D_i$ is an arc on S and that each open sub-band of U is well-situated with respect to $D_1 \cup \cdots \cup D_i$ (Step P1 for $i = 1$ and Step P2 for $i = 2, \cdots, n-1$), and then transform $\mathcal{B} \cup \mathcal{D}$ so that $B_n \cap D_n$ is an arc on S (Step P3). If $\mathcal{S}(B_i \cap D_i) = \emptyset$, then we can shrink B_i so that $B_i \cap D_i$ is an arc on S. Thus we assume that $\mathcal{S}(B_i \cap D_i) \neq \emptyset$ for each i.

Step P1: Take a look at $\mathcal{S}(B_1 \cap D_1) = \{\alpha_1, \cdots, \alpha_m\}$ and their pre-images $\{\dot{\alpha}_1, \cdots, \dot{\alpha}_m\} \in B_1^*$ and $\{\ddot{\alpha}_1, \cdots, \ddot{\alpha}_m\} \in D_1^*$, where $\dot{\alpha}_1, \cdots, \dot{\alpha}_m$ are positioned on B_1^* so that $\dot{\alpha}_k$ is closer to D_1^* than $\dot{\alpha}_{k+1}$ $(k = 1, \cdots, m-1)$ (see Figure 5 for a case that $m = 3$). We eliminate the singularities $\alpha_1, \cdots, \alpha_m$ in this order by self Δ-moves on K_1.

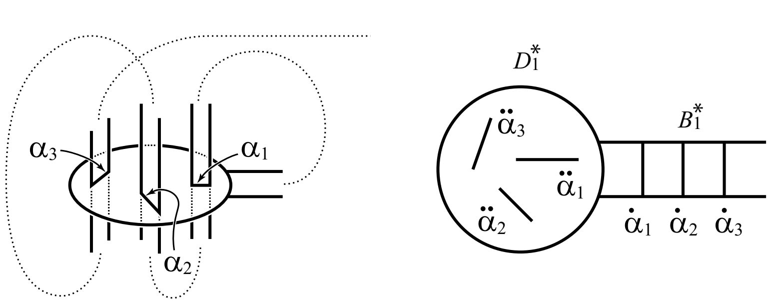

FIGURE 5

Step P1a: Let b_1 be the open sub-band of U whose ends are α_1 and $\partial B_1 \cap \partial D_1$. Since the Δ-move is an unknotting operation [4], b_1 can be transformed into the unknot by using the transformations as illustrated in Figure 4. Here an open sub-band b whose ends are on a disk D is *unknotted* if there is a disk δ such that the intersections $\delta \cap b = \partial\delta \cap b$ and $\delta \cap D = \partial\delta \cap D$ are complementary two arcs of $\partial\delta$, where the other open sub-bands may intersect with δ (see Figure 6). If an open sub-band which belongs to B_1 intersects with the disk δ for b_1, then we can remove the open sub-band out of δ by the transformations as illustrated in Figure 4. If we still have open sub-bands which intersect with the disk δ for b_1, then isotop these sub-bands out of δ and eliminate α_1 as illustrated in Figure 6. Then the band whose original ends are α_1 and α_2 has now $\partial B_1 \cap \partial D_1$ and α_2 as its ends, and thus U has one fewer components than that of the before the process. Note that the open sub-bands of U are all well-situated with respect to D_1, and that the open sub-bands of $U \cap B_1$ do not intersect with D_1.

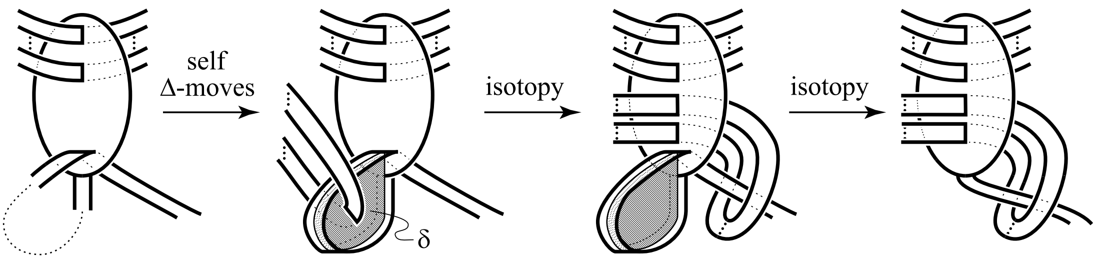

FIGURE 6

Step P1b: Assuming that we have eliminated singularities $\alpha_1, \cdots, a_{k-1}$ ($k = 2, \cdots, m$), that the open sub-bands of U are all well-situated with respect to D_1, and that the open sub-bands of $U \cap B_1$ do not intersect with D_1, take a look at α_k and let b_k be the open sub-band of U whose ends are α_k and $\partial B_1 \cap \partial D_1$. Similarly to Step P1a, transform b_k into the unknot by using the transformations as illustrated in Figure 4. However here we need to isotop the bands which do not belong to B_1 but intersect with D_1 before and after the transformations as illustrated in Figure 7 so that we can apply self Δ-moves. Then eliminating α_k similarly to Step P1a, we have that the band whose original ends are α_k and α_{k+1} has now $\partial B_1 \cap \partial D_1$ and α_{k+1} as its ends ($\alpha_{m+1} = B_1 \cap S$), and thus U has one fewer components than that of the before the process. Note that the open sub-bands of U are all well-situated with respect to D_1, and that the open sub-bands of $U \cap B_1$ do not intersect with D_1.

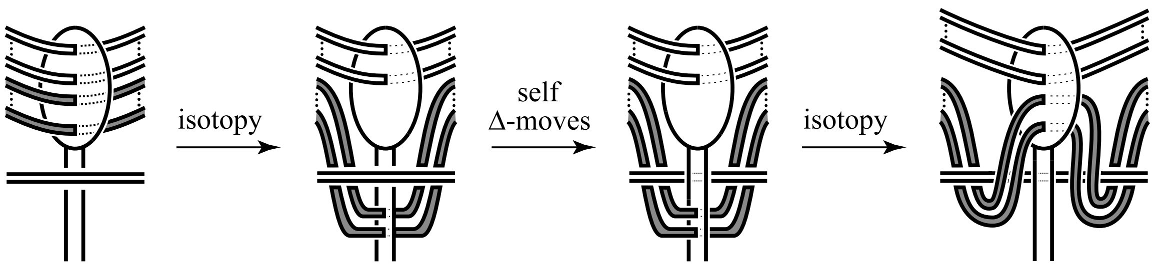

FIGURE 7

Applying the above processes, we eliminate the singularities $\alpha_1, \cdots, \alpha_m$ and thus we can shrink B_1 so that $B_1 \cap D_1 = \partial B_1 \cap \partial D_1$ is an arc on S.

Step P2: Assuming that $B_j \cap D_j = \partial B_j \cap \partial D_j$ is an arc on S $(j = 1, \cdots, k-1)$ and that each open sub-band of U is well-situated with respect to $\mathcal{D}_{k-1} = D_1 \cup \cdots \cup D_{k-1}$, transform $\mathcal{B} \cup \mathcal{D}$ so that $B_k \cap D_k = \partial B_k \cap \partial D_k$ is an arc on S and that each open sub-band of U is well-situated with respect to $D_1 \cup \cdots \cup D_k$ $(k = 2, \cdots, n-1)$. This can be done similarly to Step P1 except we remove all the sigularities of $\mathcal{S}(b \cap \mathcal{D}_{k-1})$ before transforming b into the unknot when we eliminate a singularity α, where b is the open sub-band of U whose ends are α and $\partial B_k \cap \partial D_k$. Note that each sub-band bounded by an innermost intersection pair of $\mathcal{S}(b \cap \mathcal{D}_{k-1})$ is unknotted from the construction, and thus there is a disk δ such that the intersections $\delta \cap b = \partial \delta \cap b$ and $\delta \cap D_j = \partial \delta \cap D_j$ are complementary two arcs of $\partial \delta$. Open sub-bands belonging to $B_k \cup \cdots \cup B_n$ may intersect with δ. Remove the intersections of $B_k \cap \delta$ by using the transformations as illustrated in Figure 4 or in Figure 7, and elminate the innermost intersection pair by isotoping b along δ out of D_k (this isotopy may creat new innermost intersection pairs for open sub-bands belonging to $B_{k+1} \cup \cdots \cup B_n$).

Step P3: We have that $B_i \cap D_i = \partial B_i \cap \partial D_i$ is an arc on S $(i = 1, \cdots, n-1)$ and $U \cap \mathcal{B} = U \cap B_n$, and that each open sub-band of U is well-situated with respect to $D_1 \cup \cdots \cup D_{n-1}$. Transform $\mathcal{B} \cup \mathcal{D}$ so that $B_n \cap D_n = \partial B_n \cap \partial D_n$ is an arc on S and that U is the empty set similarly to Steps P1 and P2. Then $(\ell \circ \partial \mathcal{D}) \oplus \partial \mathcal{B}$ is ambient isotopic to ℓ. Hence the proof is complete. □

The idea of the proof of Theorem 1.1 is the same as that of the proof of Proposition 2.1, and thus we prove Theorem 1.1 by referring the proof of Proposition 2.1.

Proof of Theorem 1.1. Since each D_j is a ribbon disk, we can deform $B_j \cup D_j$ into a union $(B_j^0 \cup D_j^0) \cup (B_j^1 \cup D_j^1) \cup \cdots \cup (B_j^{m_j} \cup D_j^{m_j})$ of bands $\mathcal{B}_j = B_j^0 \cup B_j^1 \cup \cdots \cup B_j^{m_j}$ and disks $\mathcal{D}_j = D_j^0 \cup D_j^1 \cup \cdots \cup D_j^{m_j}$ satisfying the following $(j = 1, \cdots, n)$ ([1]) (see Figure 8 for an example of the pre-image $\mathcal{B}_1^* \cup \mathcal{D}_1^*$):

(i) $B_j^0 = B_j$ and B_j^i connects ∂D_j^0 and ∂D_j^i $(i = 1, \cdots, m_j)$; and
(ii) Each D_j^{k*} contains only i-lines and each B_j^{k*} contains only b-lines $(k = 0, \cdots, m_j)$.

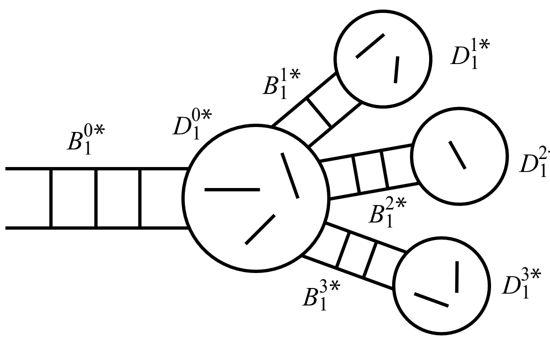

FIGURE 8

Moreover note that each $\mathcal{B}_j \cup \mathcal{D}_j$ has only self-intersections, that is, B_j^k intersects only with $\mathcal{D}_j$, since L is a separate ribbon link. Let $\ell = k_1 \cup \cdots \cup k_n$, $\ell\#L = K_1 \cup \cdots \cup K_n$, where $K_j = (k_j \circ \partial D_j) \oplus \partial B_j$. We show that $\ell\#L$ can be transformed into ℓ by applying self Δ-moves to $K_1, \cdots, K_n$ in this order.

First, for each band B_j^k $(k \neq 0)$, remove intersections of B_j^k and D_j^l with $l \neq k$ by using the transformations as illustrated in Figure 4 (a). Then we have that B_j^k (resp. D_j^0) intersects only with D_j^k (resp. B_j^0) and that D_j^k (resp. B_j^0) intersects with B_j^0 and B_j^k (resp. $\mathcal{D}_j$). Each band B_j^k and B_j^0 in int S is divided into open sub-bands by D_j^k and $\mathcal{D}_j$, respectively. Let U be the set of all such open sub-bands, that is, $U = \cup_j(((B_j^0 \cap \text{int } S) - \mathcal{D}_j) \cup \cup_{k\neq 0}((B_j^k \cap \text{int } S) - D_j^k))$. We define well-situatedness of an element of U with respect to a subset of $\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_n$ as before.

Step T1: Take a look at the singularities $\alpha_1, \cdots, \alpha_m$ of B_1^1, where pre-images $\dot{\alpha}_1, \cdots, \dot{\alpha}_m$ on B_1^{1*} are positioned so that $\dot{\alpha}_t$ is closer to D_1^{1*} than $\dot{\alpha}_{t+1}$ $(t = 1, \cdots, m-1)$. We eliminate the singularities $\alpha_1, \cdots, \alpha_m$ in this order by self Δ-moves on K_1.

Step T1a: Let b_1 be the open sub-band of U whose ends are α_1 and $\partial B_1^1 \cap \partial D_1^1$. Transform b_1 into the unknot by using the transformations as illustrated in Figure 4 as in Step P1a of the proof of Proposition 2.1. If an open sub-band which belongs to $\mathcal{B}_1$ intersects with the disk δ for b_1, then remove the open sub-band out of δ by the transformations as illustrated in Figure 4. If we still have open sub-bands which intersect with the disk δ for b_1, then isotop these sub-bands out of δ and eliminate α_1 as illustrated in Figure 6. Then the band whose original ends are α_1 and α_2 has now $\partial B_1^1 \cap \partial D_1^1$ and α_2 as its ends, and thus U has one fewer components than that of the before the process. Note that the open sub-bands of U are all well-situated with respect to D_1^1, and that the open sub-bands of $U \cap \mathcal{B}_1$ do not intersect with D_1^1.

Step T1b: Assuming that we have eliminated singularities $\alpha_1, \cdots, a_{t-1}$ $(t = 2, \cdots, m)$, that the open sub-bands of U are all well-situated with respect to D_1^1, and that the open sub-bands of $U \cap \mathcal{B}_1$ do not intersect with D_1^1, take a look at α_t and let b_t be the open sub-band of U whose ends are α_t and $\partial B_1^1 \cap \partial D_1^1$. Similarly to Step P1b of the proof of Proposition 2.1, transform b_t into the unknot by using the transformations as illustrated in Figure 4 or in Figure 7. Then eliminating α_t similarly to Step T1a, we have that the band whose original ends are α_t and α_{t+1} has now $\partial B_1^1 \cap \partial D_1^1$ and α_{t+1} as its ends $(\alpha_{m+1} = B_1^1 \cap S)$, and thus U has one fewer components than that of the before the process. Note that the open sub-bands of U are all well-situated with respect to D_1^1, and that the open sub-bands of $U \cap \mathcal{B}_1$ do not intersect with D_1^1.

Applying the above processes, there are no singularities on B_1^1. Thus we merge D_1^1 and D_1^0 into D_1^0 by shrinking B_1^1.

Step T1c: Applying Step T1a and Step T1b to $B_1^2, \cdots, B_1^{m_1}$, we have that $B_1^0 = B_1$ and $D_1^0 = D_1$, that B_1 intersects only with D_1, and that the open sub-bands of U are all well-situated with respect to D_1. Then we can elminate all the singularities of $B_1 \cap D_1$ similarly to Step T1b. Thus we can shrink B_1 so that $B_1 \cap D_1 = \partial B_1 \cap \partial D_1$ is an arc on S.

Step T2: Similarly to Step P2 and Step P3, we eliminate the singularities on B_j^k and merge D_j^k and D_j^0 into D_j^0 by shrinking B_j^k $(j = 2, \cdots, n,\ k = 1, \cdots, m_j)$, and then eliminate the singularities on B_j^0 and shrink $B_j^0 = B_j$ so that $B_j \cap D_j = \partial B_j \cap \partial D_j$ is an arc on S. Then $(\ell \circ \partial \mathcal{D}) \oplus \partial \mathcal{B}$ is ambient isotopic to ℓ. Hence the proof is complete.

□

References

[1] A. Kawauchi, T. Shibuya, and S. Suzuki, *Descriptions on surfaces in four-space. I. Normal forms*, Math. Sem. Notes Kobe Univ., **10** (1982), 75–125.

[2] K. Kobayashi, K. Kodama, and T. Shibuya, *Ribbon link and separate ribbon link*, J. Knot Theory Ramifications, **12** (2003), 105–116.

[3] S.V. Matveev, *Generalized surgeries of three-dimensional manifolds and representations of homology sphere* (in Russian), Mat. Zametki, **42** (1987), 651-656.

[4] H. Murakami and Y. Nakanishi, *On a certain move generating link-homology*, Math. Ann., **284** (1989), 75-89.

[5] Y. Nakanishi and Y. Ohyama, *Delta link homotopy for two component links. III*, J. Math. Soc. Japan, **55** (2003), 641–654.

[6] Y. Nakanishi, and T. Shibuya, *Link homotopy and quasi self delta-equivalence for links*, J. Knot Theory Ramifications, **9** (2000), 683–691.

[7] Y. Nakanishi, and T. Shibuya, and A. Yasuhara, *Self delta-equivalence of cobordant links*, Proc. Amer. Math. Soc., **134** (2006), 2465–2472.

[8] T. Shibuya, *Self* Δ*-equivalence of ribbon links*, Osaka J. Math., **33** (1996), 751–760.

[9] T. Shibuya, and A. Yasuhara, *Boundary links are self delta-equivalent to trivial links*, Math. Proc. Cambridge Philos. Soc., **143** (2007), 449–458.

[10] A. Yasuhara, *Self delta-equivalence for links whose Milnor's isotopy invariants vanish*, Trans. Amer. Math. Soc., **361** (2009), 4721–4749.