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Self Pass-Equivalence of Z_2-algebraically Split Links

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Abstract

In this paper, we study self pass-equivalence of Z_2-algebraically split links that are obtained by n-fusions of Z_2-link homologous links with n-components. In addition, we show that ribbon links with the Brunnian property are free self pass-trivial.

Keywords; knots, links, self pass-moves, fusions

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1. Introduction

All links are assumed to be ordered and oriented, and they are considered up to ambient isotopy in a 3-space $\mathbb{R}^3$. A *pass-move* is a local move on links as illustrated in Figure 1. If the four strands in the figure belong to the same component of a link, we call it a *self pass-move*. For two links ℓ and ℓ', we say that ℓ is *self pass-equivalent to* ℓ', or that ℓ and ℓ' are *self pass-equivalent* if ℓ can be transformed into ℓ' by a finite sequence of self pass-moves ([4]).

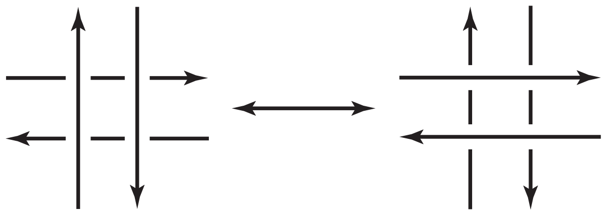

Figure 1

If two links ℓ and ℓ' are *split* in $\mathbb{R}^3$, that is, there is a 3-ball D^3 in $\mathbb{R}^3$ such that $D^3 \cap (\ell \cup \ell') = \ell$, then we denote the link $\ell \cup \ell'$ by $\ell \circ \ell'$. Let $\ell = k_1 \cup \cdots \cup k_n$ and $\ell' = k'_1 \cup \cdots \cup k'_n$ be n-component links which are split and let $\mathcal{B} = B_1 \cup \cdots \cup B_n$ be a disjoint union of disks such that $B_i \cap \ell = \partial B_i \cap k_i$ (resp. $B_i \cap \ell' = \partial B_i \cap k'_i$) is an arc for each i. Then the link $L = (\ell \circ \ell') \oplus \partial\mathcal{B}$ is called a *link obtained by an n-fusion* (or a *band sum*) of $\ell \circ \ell'$, and denoted by $\ell \underset{\mathcal{B}}{\#} \ell'$ or simply by $\ell\#\ell'$, where $\oplus$ means the homological addition. We denote the band obtained from B_i by operating p_i-full twists by $B_i^{p_i}$ (See Figure 2) and denote $(\ell \circ \ell') \oplus \partial\mathcal{B}'$ by $L_{p_1 \cdots p_n}$, where $\mathcal{B}' = B_1^{p_1} \cup \cdots \cup B_n^{p_n}$. Note that $L_{p_1 \cdots p_n}$ is link homotopic to L.

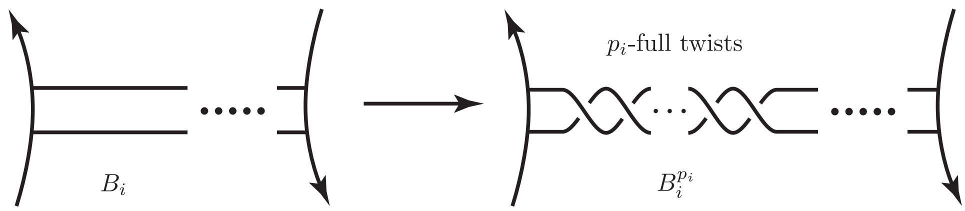

Figure 2

Two links $\ell = k_1 \cup \cdots \cup k_n$ and $\ell' = k'_1 \cup \cdots \cup k'_n$ are said to be *Z_2-link homologous* if $\mathrm{lk}(k_i, k_j) \equiv \mathrm{lk}(k'_i, k'_j) \pmod 2$ for each i and j $(i \neq j)$. If ℓ and ℓ' are Z_2-link homologous, then $\ell\#\ell' = K_1 \cup ... \cup K_n$ is *Z_2-algebraically split*, that is, $\mathrm{lk}(K_i, K_j) \equiv 0 \pmod 2$. Then we have the following.

Theorem 2.4. *Let $\ell = k_1 \cup \cdots \cup k_n$ and ℓ' be n-component Z_2-link homologous links. If $\mathrm{lk}(k_i, k_j) \equiv 1 \pmod 2$ and $p_i \equiv p_j \pmod 2$ for each i, j $(i \neq j)$, then $L = \ell\#\ell'$ and $L_{p_1 \cdots p_n}$ are self pass-equivalent.*

An n-component link $\ell = k_1 \cup \cdots \cup k_n$ is said to be *free self pass-trivial* if each k_i is self pass-equivalent to the trivial knot in $\mathbb{R}^3 - (\ell - k_i)$. A link ℓ is said to be with *Brunnian property* if $\ell - k$ is trivial for each component k of ℓ. Then we have the following.

Theorem 3.3. *If ℓ is a ribbon link with Brunnian property, then ℓ is free self pass-trivial.*

2. Self pass-equivalence of Z_2-algebraically split links

A link ℓ is said to be *proper* if $\mathrm{lk}(k, \ell - k) \equiv 0 \pmod 2$ for each component k of ℓ. If ℓ is proper, then the Arf invariant $\varphi(\ell)$ is well-defined ([3]). The following is shown in [4].

Lemma 2.1([4], Lemma 2.6)**.** *Let $\ell = k_1 \cup k_2$ and $\ell' = k_1' \cup k_2'$ be 2-component split links and $L = \ell \# \ell'$ and $L' = L_{01}$ or L_{10}. If $\mathrm{lk}(k_1, k_2) \equiv \mathrm{lk}(k_1', k_2') \equiv 1$ (mod 2), then L and L' are proper and link-homotopic, and $\varphi(L) \neq \varphi(L')$.*

Lemma 2.2. *Let $\ell = k_1 \cup k_2$ and $\ell' = k_1' \cup k_2'$ be 2-component split links and $L = \ell \# \ell'$ and $L' = L_{p_1 p_2}$. If $\mathrm{lk}(k_1, k_2) \equiv \mathrm{lk}(k_1', k_2')$ (mod 2), then we have the following;*

(i) *If $\mathrm{lk}(k_1, k_2) \equiv 0$ (mod 2), then $\varphi(L) = \varphi(L')$.*

(ii) *If $\mathrm{lk}(k_1, k_2) \equiv 1$ (mod 2), then $\varphi(L) = \varphi(L')$ if and only if $p_1 \equiv p_2$ (mod 2).*

Proof. If $\mathrm{lk}(k_1, k_2) \equiv 0 \pmod 2$, then ℓ and ℓ' are proper and thus $\ell \circ \ell'$ is proper. Hence $\varphi(L) = \varphi(L')$ ([3]). Consider the case when $\mathrm{lk}(k_1, k_2) \equiv 1 \pmod 2$. Note that a self pass-move does not change the Arf invariant of a proper link. Thus it is sufficient to consider the case when $0 \leq p_1, p_2 \leq 1$, since ± 2-full twists of a bands can be removed by a self pass-move as illustrated in Figure 3. If $p_1 \not\equiv p_2 \pmod 2$, then $(p_1, p_2) = (0, 1)$ or $(1, 0)$ and thus $\varphi(L) \neq \varphi(L')$ by Lemma 2.1. Conversely, suppose that $p_1 \equiv p_2 \pmod 2$. In this case, $(p_1, p_2) = (0, 0)$ or $(1, 1)$. In the former case, $L' = L_{00} = L$ and thus $\varphi(L) = \varphi(L')$. In the latter case, $L' = L_{11}$ and $\varphi(L) \neq \varphi(L_{01})$ and $\varphi(L_{01}) \neq \varphi(L_{11}) = \varphi(L')$ by Lemma 2.1. Hence $\varphi(L) = \varphi(L')$.

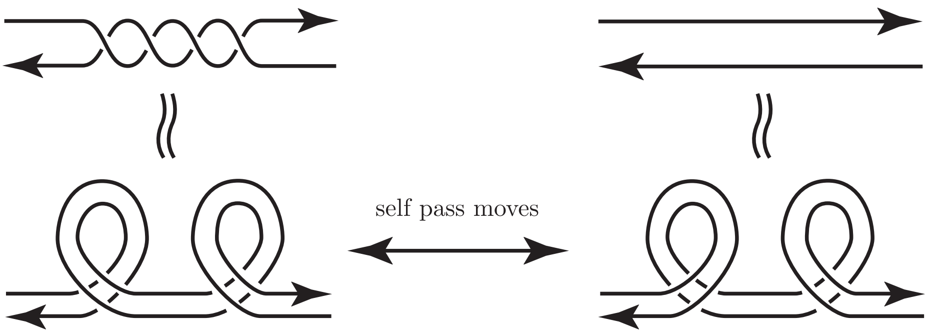

Figure 3

The following is also shown in [4].

Lemma 2.3([4], Corollary 1.3 (i))**.** *Let $L = K_1 \cup ... \cup K_n, L' = K_1' \cup \cdots \cup K_n'$ be n-component Z_2-algebraically split links. Then L and L' are self pass-equivalent if and only if L and L' are link-homotopic, $\varphi(K_i) = \varphi(K_i')$ for each i, and $\varphi(K_i \cup K_j) = \varphi(K_i' \cup K_j')$ for each i, j $(1 \leq i < j \leq n)$.*

Proof of Theorem 2.4. Since ℓ and ℓ' are Z_2-link homologous, L and $L' (= L_{p_1 \cdots p_n})$ are Z_2-algebraically split links and link-homotopic. Since each knot K_i of L and K_i' of L' are obtained by fusions of $k_i \circ k_i', \varphi(K_i) = \varphi(K_i')$. Moreover we see that $\varphi(K_i \cup K_j) = \varphi(K_i' \cup K_j')$ by the assumption of Theorem 2.4 and Lemma 2.2. Hence L and L' are self pass-equivalent by Lemma 2.3.

3. Free self pass-equivalence of ribbon links with Brunnian property.

There is a ribbon link which is not free self pass-equivalent to trivial. For example, let k be a non-trivial ribbon knot and ℓ a non-twisted pararell link of k. Then ℓ is a ribbon link. However, ℓ is not free h-trivial from Remark 1.2 (1) of [1], and thus ℓ is not free self pass-trivial. Here a link ℓ is *free h-trivial* if, for each component k of ℓ, k is homotopic to trivial in $R^3 - (\ell - k)$).

Lemma 3.1([1], Corollary 1.7)**.** *Any ribbon link with Brunnian property is free h-trivial.*

Lemma 3.2([2], Lemma 2.1)**.** *If a link $\ell = k_1 \cup \cdots \cup k_n$ is free h-trivial and $\varphi(k_i) = \varphi(k_i \cup k_j) = 0$ for each $i, j (i \neq j)$, then ℓ is free self pass-trivial.*

Proof of Theorem 3.3. If $\ell = k_1 \cup \cdots \cup k_n$ is a ribbon link, then it is easy to see that $\varphi(k_i) = \varphi(k_i \cup k_j) = 0$ for each $i, j(i \neq j)$. Hence we have the conclusion by Lemma 3.1 and Lemma 3.2.

Example 3.4. Since the link $\mathcal{L}_n$ as illustrated in Figure 4 is a ribbon link with Brunnian property, $\mathcal{L}_n$ is free self pass-equivalent to trivial for any integer n by Theorem 3.3.

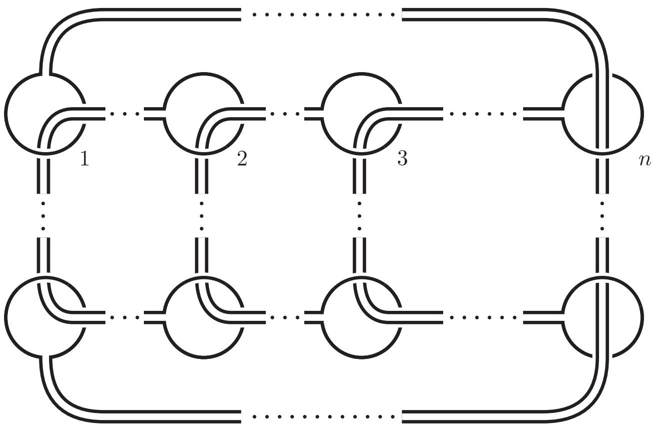

Figure 4

□

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