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# Self delta-equivalence of algebraically split links

by

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**Abstract**

It is known that any algebraically split link is $\Delta$-equivalent to a trivial link, where a link is algebraically split if the linking number of each 2-component sublink of it vanishes. However, it is also known that there is an algebraicaly split link which is not self $\Delta$-equivalent to a trivial link. In this paper, we give two sufficient conditions for an algebraically split link to be self $\Delta$-equivalent to a trivial link.

**Keywords;** knots, links, self delta-moves

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## 1. Introduction

Throughout the paper, links are tame and oriented in an oriented 3-space $\mathbb{R}^3$ and they are considered up to ambient isotopy of $\mathbb{R}^3$.

A local move on links as illustrated in Figure 1 is called a $\Delta$-*move*. If the three strands in the figure belong to the same component, then it is called a *self* $\Delta$-*move*. If a link $\ell$ can be transformed into a trivial link by a finite sequence of $\Delta$-moves (resp. self $\Delta$-moves), then we say that $\ell$ is $\Delta$-*equivalent* (resp. *self* $\Delta$-*equivalent*) *to a trivial link*.

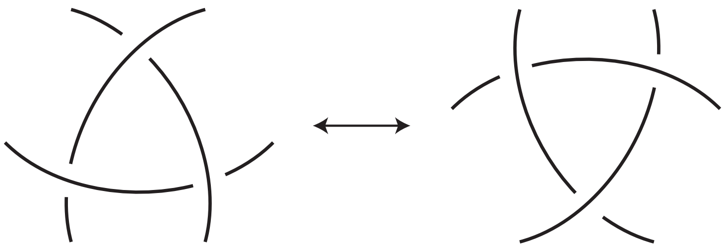

Figure 1

An $n$-component link $L = K_1 \cup \cdots \cup K_n$ is said to be *algebraically split* if the linking number, denoted by $\mathrm{lk}(K_i, K_j)$, is zero for each $i$ and $j$ $(i \neq j)$. Then the following is shown in [3].

**Proposition 1.1.** *A link $L$ is $\Delta$-equivalent to a trivial link if and only if $L$ is algebraically split.*

However, in general, algebraically split links are not self $\Delta$-equivalent to trivial links. For example, the Whitehead link and the Borromean rings are algebraically split, but not self $\Delta$-equivalent to trivial links. In this paper, we give two sufficient conditions for an algebraically split link to be self $\Delta$-equivalent to a trivial link (Theorem 3.3 and Theorem 3.7).

## 2. Self $\Delta$-equivalence of strongly related links.

In this section, we show two lemmas which we use to prove the theorems.

For two links $L(\subset \mathbb{R}^3 \times \{a\})$ and $L'(\subset \mathbb{R}^3 \times \{b\})$, $L$ is said to be *related to* $L'$ if there is a disjoint union $\mathcal{F} = F_1 \cup \cdots \cup F_m (\subset \mathbb{R}^3 \times [a, b])$ of locally flat non-singular orientable surfaces of genus 0 such that $\mathcal{F} \cap (\mathbb{R}^3 \times \{a\}) = L$, $\mathcal{F} \cap (\mathbb{R}^3 \times \{b\}) = -L'$ and $F_i \cap (\mathbb{R}^3 \times \{a\}) \neq \emptyset$, $F_i \cap (\mathbb{R}^3 \times \{b\}) \neq \emptyset$ for each $i$, where $-L'$ is the reflective inverse of $L'$ and $\mathbb{R}^3 \times [a, b] = \{(x, y, z, t) \in \mathbb{R}^4 | a \leq t \leq b\}$, $\mathbb{R}^3 \times \{c\} = \mathbb{R}^3 \times [c, c]$. Moreover if the number of components of $L$ is $m$, then we say that $L$ is *strongly related to* $L'$. Especially if each $F_i$ is an annulus, then we say that $L$ is *cobordant to* $L'$ and denote it by $L \sim L'$.

**Lemma 2.1.** *Let $L$ and $L_0$ be two links such that $L \sim L_0$. If $L_0$ is self $\Delta$-equivalent to a trivial link, then $L$ is also self $\Delta$-equivalent to a trivial link.*

*Proof.* Since the Milnor invariant is a cobordism invariant by [1], we obtain the lemma by Corollary 1.5 in [9]. □

We may consider the self $\Delta$-equivalence of links from a 4-dimensional point of view. For two links $L(\subset \mathbb{R}^3\times\{a\})$, $L'(\subset \mathbb{R}^3\times\{b\})$, $L$ is self $\Delta$-equivalent to $L'$ if and only if there is a disjoint union $\mathcal{A}=A_1\cup\cdots\cup A_n$ of level-preserving annuli in $\mathbb{R}^3\times[a,b]$ with $\mathcal{A}\cap(\mathbb{R}^3\times\{a\})=L$, $\mathcal{A}\cap(\mathbb{R}^3\times\{b\})=-L'$ which is localy flat except finite points, say $Q_1,...,Q_r$, in the interior of $\mathcal{A}$ such that $(\partial N(Q_i:\mathbb{R}^3\times[a,b]),\partial N(Q_i:\mathcal{A}))$ is a Borromean rings for each $i$. We say that $\mathcal{A}$ is *a union of level-preserving $\Delta$-annuli between $L$ and $L'$* and denote $Q_1\cup\cdots\cup Q_r$ by $\mathcal{S}(\mathcal{A})$. The following is an extention of Lemma 2.1.

**Lemma 2.2.** *Suppose that $L$ is strongly related to $L_0$. If $L_0$ is self $\Delta$-equivalent to a trivial link, then $L$ is also self $\Delta$-equivalent to a trivial link.*

*Proof.* Assume that $L$ and $L_0$ are contained in $\mathbb{R}^3\times\{0\}$ and $\mathbb{R}^3\times\{2\}$, respectively. Let $\mathcal{F}=F_1\cup\cdots\cup F_n(\subset\mathbb{R}^3\times[0,2])$ be a disjoint union of surfaces for $L$ and $L_0$ to be strongly related. Moreover as $L_0$ is self $\Delta$-equivalent to a trivial link, there is a union of level-preserving $\Delta$-annuli $\mathcal{A}=A_1\cup\cdots\cup A_n$ in $\mathbb{R}^3\times[2,3]$ between $L_0$ and a trivial link $\mathcal{O}(\subset\mathbb{R}^3\times\{3\})$. Let $P_1,...,P_r$ be the maximal points of $\mathcal{F}$ and $Q_1,...,Q_s$ the points of $\mathcal{S}(\mathcal{A})$ and $R_1,...,R_{r+s}, r+s$ points in $\mathbb{R}^3\times\{4\}$. Now we take $r+s$ level-preserving mutually disjoint arcs $\alpha_1,...,\alpha_r,\beta_1,...,\beta_s$ in $\mathbb{R}^3\times[0,4]$ such that $\partial\alpha_i=P_i\cup R_i$ and $\partial\beta_j=Q_j\cup R_{r+j}$ such that $\alpha_i\cap(\mathcal{F}\cup\mathcal{A})=P_i,\beta_j\cap(\mathcal{F}\cup\mathcal{A})=Q_j$. Then isotop $P_1\cup\cdots\cup P_r\cup Q_1\cup\cdots\cup Q_s$ to $R_1\cup\cdots\cup R_{r+s}$ along $\cup_{i,j}(\alpha_i\cup\beta_j)$. As a result, we obtain a surface $\Sigma$ in $\mathbb{R}^3\times[0,4]$ such that $\mathcal{C}(=\Sigma\cap\mathbb{R}^3\times[0,3])$ is a disjoint union of locally flat non-singular orientable surface of genus 0 with $\partial\mathcal{C}\cap\mathbb{R}^3\times\{0\}=L$ and $\partial\mathcal{C}\cap\mathbb{R}^3\times\{3\}=\{s$ Borromean rings$\}\cup\mathcal{O}'$ for a trivial link $\mathcal{O}'$.

By the similar discussion as that of proof of Lemma 1.17 in [5], we obtain a surface $\mathcal{C}_0$ by deforming $\mathcal{C}$ suitably satisfying the following:

(1) $L\sim\mathcal{L}$ for $\mathcal{L}=\mathcal{C}_0\cap\mathbb{R}^3\times\{1\}$.
(2) $\mathcal{L}(\subset\mathbb{R}^3\times\{1\})$ is self $\Delta$-equivalent to $\mathcal{L}_0(=\mathcal{C}\cap\mathbb{R}^3\times\{2\})$.
(3) $\mathcal{L}_0$ is strongly related to a trivial link $\mathcal{O}_0(=\mathcal{C}_0\cap\mathbb{R}^3\times\{3\})$.

By condition (3), $\mathcal{L}_0$ is a ribbon link, and thus $\mathcal{L}_0$ is self $\Delta$-equivalent to a trivial link by [6]. Hence $L$ is self $\Delta$-equivalent to a trivial link by conditions (1),(2) and Lemma 2.1. □

## 3. Self $\Delta$-equivalence of algebraically split links.

For the self $\Delta$-equivalence of boundary links [8], the following is proved in [7].

**Lemma 3.1.** ([7, Theorem 1·1]) *Any boundary link is self $\Delta$-equivalent to a trivial link.*

The singularity as illustrated in Figure 2(a) (resp. 2(b)) is called an arc of *ribbon-type* (resp. an arc of *clasp-type*). Let $F$ be an orientable surface (or a union of disks) such that $\mathcal{S}(F)$, the set of singularities of $F$, does not have an arc of clasp-type. Assume that $T(F)$, the set of triple points of $F$, is not empty. A point $P$ of $T(F)$ is called one of type I (resp. II) if the three points $P^*$, $P'^*$, $P''^*$ of the pre-image of $P$ are those as illustrated in Figure 3(a)(resp. 3(b)). The set of points of type I (resp. II) of $T(F)$ is denoted by $T_{\mathrm{I}}(F)$ (resp. $T_{\mathrm{II}}(F)$). Then we have that $T(F)=T_{\mathrm{I}}(F)\cup T_{\mathrm{II}}(F)$. The following is shown in [2]. Here we give an alternative proof.

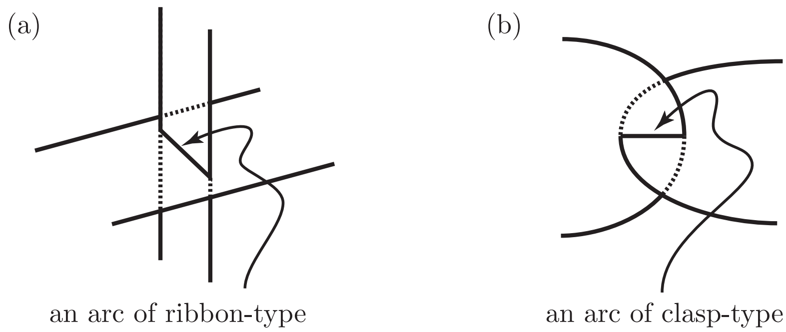


Figure 2

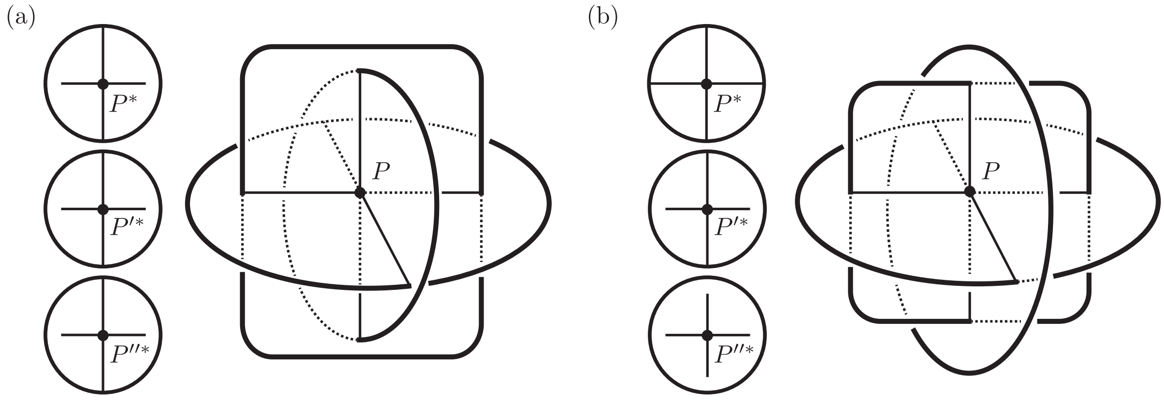


Figure 3

**Lemma 3.2.** ([2, Proposition 2.1]) *An $n$-component link $\ell = K_1 \cup \cdots \cup K_n$ is an algebraically split link if and only if there is a union $\mathcal{F} = F_1 \cup \cdots \cup F_n$ of non-singular orientable surfaces in $\mathbb{R}^3$ with $\partial \mathcal{F} = \ell$, $\partial F_i = K_i$ such that $\mathcal{S}(\mathcal{F})$ consists of mutually disjoint simple arcs of ribbon-type.*

*Proof.* If there is a surface $\mathcal{F} = F_1 \cup \cdots \cup F_n$ satisfying the above conditions, we easily see that $\ell$ is algebraically split.

Conversely, suppose that $\ell$ is algebraically split. Since $\mathrm{lk}(K_1, K_i) = 0$ for $i \geq 2$, there is a non-singular orientable surface $F_1$ with $\partial F_1 = K_1$ such that $F_1 \cap (\ell - K_1) = \emptyset$. Next as $\mathrm{lk}(K_2, K_i) = 0$ for $i \geq 3$, there is a non-singular orientable surface $F_2$ with $\partial F_2 = K_2$ such that $F_2 \cap (\ell - K_1 - K_2) = \emptyset$. Since $F_1 \cap K_2 = \emptyset, F_1 \cap F_2$ does not have an arc of clasp-type. If $F_1 \cap F_2$ contains a loop, it can be easily transformed into an arc of ribbon-type by deforming $F_1$ slightly. Hence $F_1 \cap F_2$ consists of mutually disjoint simple arcs of ribbon-type such that, for each arc $\alpha$ of $F_1 \cap F_2$, the *b-line* $\alpha^*$ of the pre-image of $\alpha$ is contained in $F_1^*$, namely $\partial \alpha^* \subset K_1^*$, for $F_1 \cap K_2 = \emptyset$, where $X^*$ means the pre-image of $X$.

By the same discussion as above, we obtain a non-singular orientable surface $F_3$ with $\partial F_3 = K_3$ such that $F_3 \cap (\ell - K_1 - K_2 - K_3) = \emptyset$ and $F_r \cap F_3$, if not empty for $r = 1, 2$, consists of mutually disjoint simple arcs of ribbon-type and the $b$-line $\alpha^*$ of the pre-image of $\alpha$ is contained in $(F_1 \cup F_2)^*$ for each $\alpha$ of $F_r \cap F_3$. Thus if $F_1 \cap F_2 \cap F_3$ is not empty, then there are three arcs, say $\alpha, \beta, \gamma$, of ribbon-type such that $\alpha \subset F_1 \cap F_2, \beta \subset F_1 \cap F_3$ and $\gamma \subset F_2 \cap F_3$ and $\alpha \cap \beta \cap \gamma$ contains a point, say $P$. By the construction of $F_1, F_2$ and $F_3$, $P$ is a triple point of type II and three points $P^*, P'^*$ and $P''^*$ are contained in $F_1^*, F_2^*$ and $F_3^*$ respectively (see Figure 4(a)).

Now deform $N(P : F_3)$ along an arc of $\alpha - P$ towards to a point of $\partial\alpha$ (see Figure 4(b)). As a result, we obtain a non-singular orientable surface $F_3'$ by $F_3$ such that $T(F_1 \cup F_2 \cup F_3') = T(F_1 \cup F_2 \cup F_3) - \{P\}$. By doing the above, any two arcs of ribbon-type of $F_i \cap F_j, i, j = 1, 2, 3 (i \neq j)$ are mutually disjoint and simple.

By performing the above discussion successively, we obtain a surface $\mathcal{F} = F_1 \cup \cdots \cup F_n$ satisfying the conditions of Lemma 3.2. □

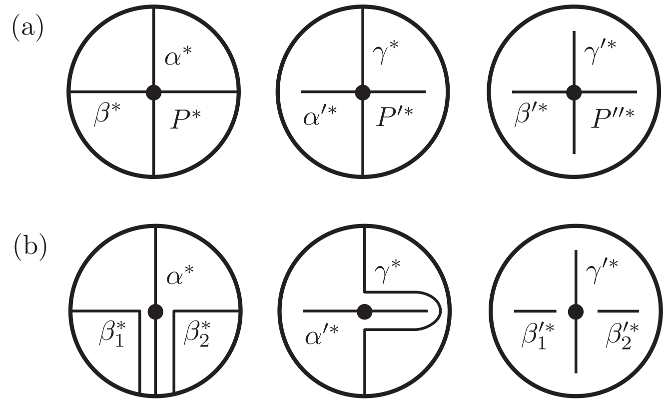


FIGURE 4

**Theorem 3.3.** *Let $\ell$ be an algebraically split link and $\mathcal{F} = F_1 \cup \cdots \cup F_n$ a surface in Lemma 3.2. If, for each arc $\alpha$ of ribbon-type of $F_i \cap F_j$ $(i, j = 1, ..., n)$, $F_i - \alpha$ (or $F_j - \alpha$) is disconnected, then $\ell$ is self $\Delta$-equivalent to a trivial link.*

*Proof.* Let $\alpha$ be an arc of ribbon-type of $F_i \cap F_j$ such that $F_i - \alpha$ is disconnected. By performing the fission of $\ell$ along $\alpha$, we obtain a link $\ell'$. Namely $\ell' = \ell \oplus \partial N(\alpha : F_i)$, where $\oplus$ means the homological addition. By performing the above fission to each $\alpha$ of $F_i \cap F_j$ for $i, j = 1, ..., n$, we obtain a link $L(= \ell \oplus (\cup_\alpha \partial N(\alpha : F_i)))$ and a union $\tilde{\mathcal{F}} = \cup_{i=1}^{n} \cup_{j=1}^{m_i} F_{ij}$ of mutually disjoint non-singular orientable surfaces with $\partial\tilde{\mathcal{F}} = L$ from $\ell$ and $\mathcal{F}$, respectively.

By the construction of $\tilde{\mathcal{F}}$, we easily see that $\ell$ is strongly related to $L$ and that $L$ is a boundary link. Hence $L$ is self $\Delta$-equivalent to a trivial link by Lemma 3.1, and thus $\ell$ is self $\Delta$-equivalent to a trivial link by Lemma 2.2. □

A link $\ell = K_1 \cup \cdots \cup K_n$ is called *a weakly $\Delta$-split link* if there is a union $\mathcal{D} = D_1 \cup \cdots \cup D_n$ of claspless disks in $\mathbb{R}^3$ with $\partial\mathcal{D} = \ell$, $\partial D_i = K_i$. Especially if $D_i \cap D_j = \emptyset$ for each $i, j (i \neq j), \ell$ is called *a $\Delta$-split link*. Since free self $\Delta$-equivalence implies self $\Delta$-equivalence, we obtain the following by Theorem 1.4 in [4].

**Lemma 3.4.** *Any $\Delta$-split link is self $\Delta$-equivalent to a trivial link.*

Suppose that $\ell$ is a weakly $\Delta$-split link and $\mathcal{D}$ the above. Then any point of $T(\mathcal{D})$ is contained in one of $\mathcal{S}(D_i), \mathcal{S}(D_i) \cap D_j$ and $D_i \cap D_j \cap D_h$ for distinct integers $i, j$ and $h$.

**Lemma 3.5.** *A link $\ell$ is a weakly $\Delta$-split link if and only if it is an algebraically split link.*

*Proof.* Suppose that $\ell$ is a weakly $\Delta$-split link. Then there is a union $\mathcal{D} = \cup_i D_i, \partial\mathcal{D} = \ell$ of claspless disks in $\mathbb{R}^3$, namely each of $D_i \cap D_j$ consists of arcs of ribbon-type or loops for $i \neq j$. Hence $\mathrm{lk}(K_i, K_j) = I(D_i, K_j) = 0$ for $K_i = \partial D_i$, where $I(x, y)$ means the intersection number of $x$ and $y$. Therefore $\ell$ is algebraically split.

Conversely if $\ell$ is algebraically split, $\ell$ is obtained by a fusion of a trivial link $\mathcal{O}$ and several copies of the Borromeans rings $\mathbf{B}$ which are split from $\mathcal{O}$ by Proposition 1.1. For each of $\mathbf{B}$, we span a union of claspless disks as illustrated in Figure 3 (a). Hence $\ell$ is a weakly $\Delta$-split link. □

**Lemma 3.6.** *A link $\ell$ is self $\Delta$-equivalent to a trivial link if and only if there is a union $\mathcal{D} = \cup_i D_i$ of claspless disks with $\partial\mathcal{D} = \ell$ such that $T_{\mathrm{I}}(\mathcal{D}) = \cup_i T_{\mathrm{I}}(D_i)$ and $T_{\mathrm{II}}(\mathcal{D}) = \emptyset$.*

*Proof.* If $\ell$ is self $\Delta$-equivalent to a trivial link, then $\ell$ is obtained by a fusion of a trivial link $\mathcal{O}$ and several copies of the Borromean rings satisfying that, for each Borromean rings $B$, there are 3 bands $b_1, b_2$ and $b_3$ of fusion of $B$ and $\mathcal{O}$ such that $b_i \cap B \neq \emptyset$ and $b_i \cap O \neq \emptyset$ for $i = 1, 2, 3$ and some component $O$ of $\mathcal{O}$. Hence, by spanning the claspless disks as illustrated in Figure 3(a) to each Borromean rings, we obtain the necessity.

Conversely, suppose that there is a union $\mathcal{D} = \cup_i D_i$ satisfying the conditions of Lemma 3.6. If $D_i \cap D_j = \emptyset$ for each $i, j (i \neq j), \ell$ is a $\Delta$-split link and so $\ell$ is self $\Delta$-equivalent to a trivial link by Lemma 3.4. Next suppose that $D_i \cap D_j \neq \emptyset$ for $i \neq j$. Let $\alpha$ be an arc of ribbon-type of $D_i \cap D_j$. Then $\alpha$ is simple (i.e. no self intersections) and $\alpha \cap T(\mathcal{D}) = \emptyset$ because $T_{\mathrm{I}}(\mathcal{D}) = \cup_i T_{\mathrm{I}}(D_i)$ and $T_{\mathrm{II}}(D) = \emptyset$ . Assume that the $b$-line of the pre-image of $\alpha$ is contained in the pre-image of $D_i$. Now perform the fission along each such an arc $\alpha$ on $D_i$, we may obtain a link $L$ and a union of claspless disks $\mathcal{E} = \cup_i E_i$ from $\ell$ and $\mathcal{D}$, respectively such that $\mathcal{E}$ is a disjoint union, namely $\mathcal{E} = cl(\mathcal{D} - \cup_\alpha N(\alpha : D_i))$ and $L = \partial\mathcal{E}$. Hence $\ell$ is strongly related to $L$. Since $\mathcal{E}$ is a disjoint union of claspless disks, $L$ is a $\Delta$-split link. Therefore $\ell$ is self $\Delta$-equivalent to a trivial link by Lemmas 2.2 and 3.4. □

**Theorem 3.7.** *If $\ell$ is an algebraically split link and $\mathcal{D} = \cup_i D_i$ is a union of claspless disks with $\partial\mathcal{D} = \ell$ such that $T_{\mathrm{I}}(\mathcal{D}) = \cup_i T_{\mathrm{I}}(D_i)$, then $\ell$ is self $\Delta$-equivalent to a trivial link.*

*Proof.* Let $f$ be an immersion of $\mathcal{D}^*$ into $\mathbb{R}^3$. For each point $P$ of $T_{\mathrm{II}}(\mathcal{D})$, $P^*$ is the point of $f^{-1}(P)$ as illustrated in Figure 3(b). Then $\mathcal{E} = cl(\mathcal{D} - \cup_P f(N(P^* : \mathcal{D}^*)))$ is a union of perforated claspless disks with $\partial\mathcal{E} = \ell \circ \mathcal{O}$, where $\mathcal{O}(= \partial(\cup_P f(N(P^* : \mathcal{D}^*))))$ is a trivial link and $\circ$ means that $\ell$ is split from $\mathcal{O}$.

By the assumption of Theorem 3.7 and the construction of $\mathcal{E}$, $T_{\mathrm{I}}(\mathcal{E}) = T_{\mathrm{I}}(\mathcal{D}) = \cup_i T_{\mathrm{I}}(D_i) = \cup_i T_{\mathrm{I}}(E_i)$ and $T_{\mathrm{II}}(\mathcal{E}) = \emptyset$, where $E_i = cl(D_i - \cup_P f(N(P^* : \mathcal{D}^*)))$. Since $T_{\mathrm{II}}(\mathcal{E}) = \emptyset$, each $i$-line of $f^{-1}(\mathcal{S}(\mathcal{E}))$ is simple and any two $i$-lines do not intersect to each other on $\mathcal{E}^*(= f^{-1}(\mathcal{E}))$. Hence there is a disjoint union $\beta^* = \cup_j \beta_j^*$ of simple arcs on $\mathcal{E}^*$ such that $\beta_j^*$ connects a point of $\ell^*$ and one of $O_j^*$ such that $\beta_j^* \cap \{i\text{-lines of } f^{-1}(\mathcal{S}(\mathcal{E}))\} = \emptyset$, where $\mathcal{O} = \cup_j O_j$.

Let $\mathcal{F} = cl(\mathcal{E} - f(N(\beta^* : \mathcal{E}^*)))(= F_1 \cup \cdots \cup F_n)$. Then $\mathcal{F}$ is a union of claspless disks such that $T_{\mathrm{I}}(\mathcal{F}) = T_{\mathrm{I}}(\mathcal{E}) = \cup_i T_{\mathrm{I}}(E_i) = \cup_i T_{\mathrm{I}}(F_i)$ and $T_{\mathrm{II}}(\mathcal{F}) = \emptyset$ by the choice of $\beta$. Hence $L(= \partial\mathcal{F})$ is self $\Delta$-equivalent to a trivial link by Lemma 3.6. Moreover as $L$ is obtained by a fusion of $\ell \circ \mathcal{O}$, $\ell$ is cobordant to $L$. Therefore $\ell$ is self $\Delta$-equivalent to a trivial link by Lemma 2.1. □

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