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On attendant graph of simple ribbon moves for links

by

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Abstract

In [2], simple ribbon moves for links (of type I) are defined and a sufficient condition for the moves to change link types is given in terms of links associated with the moves. In this paper, we give a necessary and sufficient condition for the moves to change link types in terms of graphs associated with the moves.

Keywords; knots, links

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1. Introduction

All knots and links are assumed to be ordered and oriented, and they are considered up to ambient isotopy in an oriented 3-sphere S^3.

Local moves, called *simple ribbon moves* for links, are defined and studied in [2] and [3]. The definition of a simple ribbon move is the following.

Let H be a 3-ball in S^3 and $\mathcal{D} = D_1 \cup \cdots \cup D_m$ (resp. $\mathcal{B} = B_1 \cup \cdots \cup B_m$) be a union of mutually disjoint disks (resp. bands) in int$H$ (resp. H) satisfying the following:

(1) each band B_i has its ends on ∂H and ∂D_i; and

(2) $\mathcal{S}(B_i \cup \mathcal{D}) = \mathcal{S}(B_i \cup D_{\pi(i)}) = \{\text{an arc of ribbon type}\}$, where $\mathcal{S}(X)$ means the set of singularities of X and π is a permutation on $\{1, 2, \ldots, m\}$.

Then we call $\cup_i(\partial(B_i \cup D_i) - \mathrm{int}(B_i \cap \partial H))$ an *SR-tangle* and denote it by $\mathcal{T}$.

Let ℓ be a link in $S^3 - \mathrm{int}H$ such that $\ell \cap \partial H$ consists of arcs. Take an SR-tangle $\mathcal{T}$ such that $\mathcal{B} \cap \partial H = \ell \cap \partial H$. Then let L be the link obtained from ℓ by substituting $\mathcal{T}$ for $\ell \cap \partial H$. We call the transformation either from ℓ to L or from L to ℓ a *simple ribbon-move* or an *SR-move*, and H (resp. $\mathcal{T}$) the *associated* 3-*ball* (*resp. tangle*) *of* the SR-move. The transformation from ℓ to L (resp. from L to ℓ) is called an SR^+-*move* (resp. SR^--*move*).

Since every permutation is a product of cyclic permutations, we rename the index of the bands and disks as

$$\mathcal{B} = \cup_{k=1}^n \mathcal{B}^k = \cup_{k=1}^n (\cup_{i=1}^{m_k} B_i^k) \quad \text{and} \quad \mathcal{D} = \cup_{k=1}^n \mathcal{D}^k = \cup_{k=1}^n (\cup_{i=1}^{m_k} D_i^k), \quad \text{where}$$

(1) $1 \leq m_1 \leq m_2 \leq \cdots \leq m_n$;

(2) each band B_i^k has its ends on ∂H and ∂D_i^k; and

(3) $\mathcal{S}(B_i^k \cup \mathcal{D}) = \mathcal{S}(B_i^k \cup D_{i+1}^k) = \{\text{an arc of ribbon type}\}$.

In Condition (3), the lower indices are considered modulo m_k. For an SR-tangle $\mathcal{T}$, we call $\cup_{i=1}^{m_k}(\partial(B_i^k \cup D_i^k) - \mathrm{int}(B_i^k \cap \partial H))$ the (k-th) *component* of the SR-move or of the SR-tangle, denote it by $\mathcal{T}^k$, and call m_k the *index* of the component. The *type* of the SR-move or of the SR-tangle is the ordered set $(m_1, m_2, \ldots, m_n)$ of the indices. If the index of each component is 1 (resp. no less than 2), then we say that the SR-move or the SR-tangle has type I or type I_n (resp. type II or type II_n). In this note, we consider only SR-moves of type I.

Consider an SR-move of type I_n on a link ℓ. Then each band $B^k = B_1^k$ can be regarded as $(b_1^k \cup b_2^k) \times [-1, 1]$, where b_1^k (resp. b_2^k) is an arc with ends on intD^k and on ∂D^k (resp. ℓ) (see Figure 1). Let c^k be an arc on D^k with $\partial c^k = \partial b_1^k$. We call $\mathcal{L}_{\mathcal{T}} = \cup \alpha_k = \cup(b_1^k \cup c^k)$ the *attendant link* of the SR-move or of the SR-tangle. Put vertices v_1^k (resp. v_2^k) on $b_1^k \cap b_2^k$ (resp. $b_2^k \cap \partial H$) and take a path $P = v_2^1 e^1 v_2^2 e^2 \cdots e^{n-1} v_2^n$ on ∂H. We call $\mathcal{G}_{\mathcal{T}} = (\cup \beta_k) \cup P = \cup(\alpha_k \cup v_1^k \cup b_2^k) \cup P$ the *attendant graph* of the SR-move or of the SR-tangle. Note that $\mathcal{L}_{\mathcal{T}}$ and $\mathcal{G}_{\mathcal{T}}$ are well-defined up to ambient isotopy in S^3.

If there is a union $\mathcal{M} = M_1 \cup \cdots \cup M_n$ ($n \geq 2$) of mutually disjoint non-singular 3-balls in H such that $M_k \cap \mathcal{L}_{\mathcal{T}} = \alpha_k$ for each k, then we say that $\mathcal{L}_{\mathcal{T}}$ is *completely split*. The following is shown in [2].

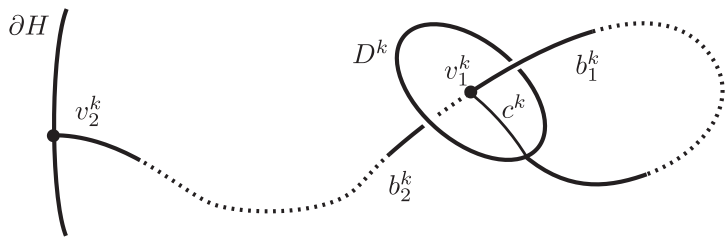

FIGURE 1

Proposition 1.1. ([2, Corollary 1.15]) *Let L be a link which can be transformed into a non-split link ℓ by an SR^--move of type* I_n $(n \geq 2)$*. If its attendant link is not completely split, then L is not ambient isotopic to ℓ.*

Remark 1.2. An SR-move of type I_1 does not change a link type (see [2]).

An attendant graph $\mathcal{G}_{\mathcal{T}} = (\beta_1 \cup \cdots \cup \beta_n) \cup P$ $(n \geq 2)$ is said to be *separable* if there is a non-singular disk E proper in $H - (\mathcal{G}_{\mathcal{T}} - P)$ such that each component of $H - E$ contains a component of $\cup\beta_k$. Then E is called *a separating disk* of $\mathcal{G}_{\mathcal{T}}$. Moreover $\mathcal{G}_{\mathcal{T}}$ is said to be *completely separable* if there is a union $\mathcal{E} = E_1 \cup \cdots \cup E_{n-1}$ of mutually disjoint non-singular disks proper in $H - (\mathcal{G}_{\mathcal{T}} - P)$ such that each component of $H - \mathcal{E}$ contains a component of $\cup\beta_k$.

Theorem 1.3. *Let L be a link which can be transformed into a non-split link ℓ by an SR^--move of type* I_n $(n \geq 2)$*. Then L is ambient isotopic to ℓ if and only if its attendant graph is completely separable.*

Remark 1.4. Since $\mathcal{G}_{\mathcal{T}}$ contains $\mathcal{L}_{\mathcal{T}}$, we have that $\mathcal{L}_{\mathcal{T}}$ is completely split if $\mathcal{G}_{\mathcal{T}}$ is completely separable (the converse does not hold. See the knot K of Remark 1.7). Thus Theorem 1.3 extends Proposition 1.1.

Remark 1.5. If ℓ is a split link, the only if part of the theorem is not true. For example, let ℓ be a 2-component trivial link $\mathcal{O}$, and L the link as illustrated in Figure 2. Then L can be transformed into $\mathcal{O}$ by an SR-move of type I_2. Since the attendant graph $\mathcal{G}_{\mathcal{T}}$ contains the Hopf link, $\mathcal{G}_{\mathcal{T}}$ is not completely separable. However, L is ambient isotopic to $\mathcal{O}$.

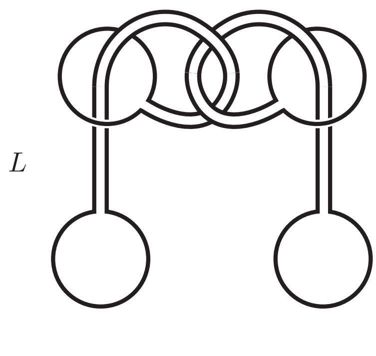

FIGURE 2

Consider an SR-tangle $\mathcal{T}$ of type I_n and its attendant graph $\mathcal{G}_\mathcal{T}$. Take a 2-component sublink $\alpha_i \cup \alpha_j$ of $\mathcal{L}_\mathcal{T} \subset \mathcal{G}_\mathcal{T}$. We define κ_{ij} of $\mathcal{T}$ $(i < j)$ as the link obtained from $\alpha_i \cup \alpha_j$ by the band-sum along the band $B^i_{1,2} \cup B_{ij} \cup B^j_{1,2}$, where $B^i_{1,2}$ (resp. $B^j_{1,2}$) is the subband of $B^i_1 = B^i$ (resp. $B^j_1 = B^j$) which is $b^i_2 \times [-1,1]$ (resp. $b^j_2 \times [-1,1]$) and B_{ij} is the band in ∂H which is $v^i_2 e^i \cdots e^{j-1} v^j_2 \times [-1,1]$ and whose ends are $B^i_{1,2} \cap \partial H$ and $B^j_{1,2} \cap \partial H$.

Proposition 1.6. *If $\mathcal{G}_\mathcal{T}$ is completely separable, then κ_{ij} is ambient isotopic to the connected sum of α_i and α_j for any i and j.*

Proof. Since $\mathcal{G}_\mathcal{T}$ is completely separable, there is a non-singular disk E proper in $H - (\mathcal{G}_\mathcal{T} - P)$ such that each component of $H - E$ contains α_i or α_j. Then we can have a 2-sphere $E \cup E'$ which intersects κ_{ij} in two points, where E' is the closure of a component of $\partial H - \partial E$. Thus κ_{ij} is ambient isotopic to the connected sum of α_i and α_j. □

Remark 1.7. The knot K as illustrated in Figure 3 can be transformed into the trivial knot by an SR-move of type I_2. Since κ_{12} is 6_1 and the connected sum of α_1 and α_2 is a trivial link, $\mathcal{G}_\mathcal{T}$ is not completely separable by the above proposition. Hence K is not trivial by Theorem 1.3. Note that we cannot decide if K is trivial or not by using Proposition 1.1, since its attendant link $\mathcal{L}_\mathcal{T} = \alpha_1 \cup \alpha_2$ is trivial.

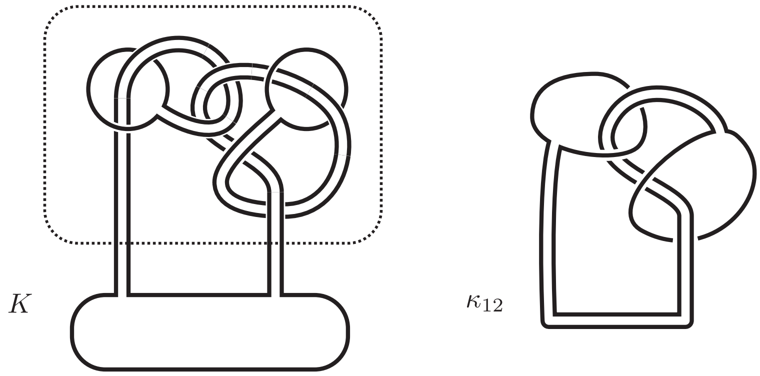

Figure 3

2. Proof of Theorem 1.3.

Proof of Theorem 1.3. First of all, we prove the if part. Suppose that the attendant graph $\mathcal{G}_\mathcal{T}(= \beta_1 \cup \cdots \cup \beta_n)$ of the SR^--move is completely separable. Then there is a union $\mathcal{E}(= E_1 \cup \cdots \cup E_{n-1})$ of mutually disjoint non-singular disks proper in $H - \mathcal{G}_\mathcal{T}$ such that each component of $H - \mathcal{E}$ contains a component of $\mathcal{G}_\mathcal{T}$. Take a neighborhood $N(\mathcal{G}_\mathcal{T})$ of $\mathcal{G}_\mathcal{T}$ in $H - \mathcal{E}$ and isotop $\mathcal{T}$ along $\mathcal{B} \cup \mathcal{D}$ so to be in $N(\mathcal{G}_\mathcal{T})$. Then we have that $\mathcal{T}$ is completely separable. Thus L is ambient isotopic to ℓ from Corollary 1.14 and Corollary 1.5 of [2].

Next we prove the only if part. Since L is ambient isotopic to ℓ, there is a union $\mathcal{F}(= F_1 \cup \cdots \cup F_{n-1})$ of mutually disjoint non-singular disks proper in $H - (\mathcal{B} \cup \mathcal{D})$ such that each component of $H - \mathcal{F}$ contains a component of $\mathcal{B} \cup \mathcal{D}$ from Corollary 1.5 and Theorem 1.12 of [2]. This implies that $\mathcal{G}_\mathcal{T}$ is completely separable, since $\mathcal{G}_\mathcal{T}$ is contained in $\mathcal{B} \cup \mathcal{D}$. □

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